Mathematical Reviews

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MATHEMATICAL REVIEWS

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Journal references in Mathematical Reviews are now given in the following form: J. Brobdingnag. Acad. Sci. (7) **4 (82)** (1952/53), no. 3, 17–42 (1954), where after the abbreviated title one has: (series number) volume number (volume number in first series if given) (nominal date), issue number if necessary, first page–last page (imprint date). In case only one date is given, this will usually be interpreted as the nominal date and printed immediately after the volume number (this is a change from past practice in Mathematical Reviews where a single date has been interpreted as the imprint date). If no volume number is given, the year will be used in its place. The symbol ★ precedes the title of a book or other non-periodical which is being reviewed as a whole.

References to reviews in Mathematical Reviews before volume 20 (1959) are by volume and page number, as MR **19**, 532: from volume 20 on, by volume and review number, as MR **20** #4387. Reviews reprinted from Applied Mechanics Reviews, Referativnyĭ Žurnal, or Zentralblatt für Mathematik are identified in parentheses following the reviewer's name by AMR, RŽMat (or RŽMeh, RŽAstr. Geod.), Zbl, respectively.

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BIBLIOGRAPHICAL NOTES

3635:

Izvestiya Vysših Učebnyh Zavedeniĭ. Matematika.

A bimonthly journal printing short articles "containing detailed presentation of original results in the field of mathematics, as well as works on mechanics having mathematical interest". The editor-in-chief is A. P. Norden. No. 1 of the year 1958 was whole number 2. Orders, at 10 rubles an issue, may be sent to Izdatel'stvo Gosudarstvennogo Universiteta imeni V. I. Ul'yanova-Lenina, ul. V. I. Lenina, 18, Kazan', U.S.S.R.

3636:

Journal of Mathematical Analysis and Applications.

Annual volumes of four issues each; Vol. 1, no. 1 is dated June 1960. Subscriptions may be ordered from the publishers, Academic Press, Inc., at 111 Fifth Ave., New York 3, N.Y., or at 17 Old Queen St., London, S.W.1. The price of Vol. 1 is $16.00. The journal "will publish carefully selected mathematical papers treating classical analysis and its manifold applications". Manuscripts may be accepted by any one of the Associate Editors or by the Editor, Richard Bellman, The RAND Corporation, 1700 Main St., Santa Monica, Calif.

3637:

Sibirskiĭ Matematičeskiĭ Žurnal.

This bimonthly journal begins with Vol. 1, no. 1, dated May–June 1960; issues are priced at 15 rubles. The address is ul. Mičurina, d. 23, Novosibirsk, U.S.S.R. Edited by A. I. Mal'cev and others, the journal "publishes original articles by Soviet and foreign authors, containing new results in any branch of mathematics".

3638:

Mathematica.

Published by the Societatea de Ştiinţe Matematice şi Fizice din R.P.R. Filiala Cluj, as a continuation of the former series under the same title 1929–1948. Volume 1 of the series (24 of the complete series), fascicule 1, is dated 1959. The journal will appear annually in two fascicules of about 160 pages each. It will contain mathematical articles in Romanian, Russian, French, English, German and Italian. Manuscripts should be sent to Tiberiu Popoviciu, Institut de Calcul, Str. Republicii 37, Cluj, Romania.

3639:

Annales de l'Association Internationale pour le Calcul Analogique. (Title also in English.)

Publication trimestrielle. Volume I, no. 1 is dated January 1959. Subscriptions (250 francs belges per year in Belgium; 350 francs belges elsewhere) are to be ordered from Ed. Hianné, 3, rue Ravenstein, Brussels 1, Belgium.

3640:

Progress in Astronautics and Rocketry.

A book series sponsored by the American Rocket Society, New York, under the editorship of Martin Summerfield. Vol. 1 is Solid propellant rocket research [Academic Press, New York-London, 1960]. Volumes in preparation are: 2, Liquid rockets and propellants; 3–4, Space power systems; 5, Electrostatic propulsion.

3641:

Soviet Mathematics—Doklady.

Contains the entire Mathematics section of Dokl. Akad. Nauk SSSR in English translation. Each volume is in six issues; each issue corresponds to one volume (i.e., two months) of the Soviet Doklady. Vol. 1, no. 1, dated 1960, corresponds to Vol. 130 of the original. Subscriptions may be ordered from the American Mathematical Society at these rates: U.S., $17.50 per volume; foreign, $20.00 per volume; single issues, $5.00.

GENERAL

3642:

Filkorn, Vojtech. Die wissenschaftliche Analyse. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe 8 (1958/59), 445–450.

3643:

★American Mathematical Society Translations. Series 2, Vol. 12. American Mathematical Society, Providence, R.I., 1959. iv+342 pp. $4.60.

This volume contains translations of twelve papers from Russian periodicals, on analysis [#3947 a–b, 3779], eigenvalue estimation [#3956 a–b, 3737, 3955], information theory [#4573, 4574 a–b], and Lie groups [#4073, 4074].

3644:
★**American Mathematical Society Translations. Series 2, Vol. 13.** American Mathematical Society, Providence, R.I., 1960. iii+346 pp. $5.30.
This volume consists of six translations of Russian papers on spectral theory of non-self-adjoint operators [#3981 a–b, 3983, #3977, 3984, 3982], and three on number theory [#3711, 3731, 3720].

3645:
★**American Mathematical Society Translations. Series 2, Vol. 14.** American Mathematical Society, Providence, R.I., 1960. iv+332 pp. $5.30.
Eleven articles: #3771, 3766, 3769, 3999, 4072, 3937, 3854, 3884, 3954, 3923, 3936.

3646:
Kelley, John L. ★**Introduction to modern algebra.** The University Series in Undergraduate Mathematics. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London-New York, 1960. x+338 pp. $2.75.
From the preface: "The text is designed for a first-year course in a liberal arts college program, or for a course for high school students in an accelerated program. Many recommendations of professional groups concerned with the mathematical curriculum have been embodied in the text, and it may, therefore, be useful for teachers of high school mathematics." {It is also the text of the author's nationally televised course.} "Some of the problem lists are quite long, and many of the problems are substantial mathematical theorems. The material which is covered in the text comprised a completely unorthodox course in college algebra." The last three chapters (more than half the book) are entitled: The complex plane, Vector geometry, Matrix algebra. Vector spaces have at most 3 dimensions (except in a digression on quaternions).

3647:
★**Англо-Русский словарь по радиоэлектронике. [English-Russian dictionary in radioelectronics.]** Compiled by N. I. Dozorov. Voennoe Izdat. Ministr. Oboron. SSSR, Moscow, 1959. 535 pp. 16.90 r.
From the preface: "The dictionary contains about 20,000 terms from radio engineering, electronics and related fields. More attention has been paid to military applications, radar, anti-radar, radio communication, technical aspects of infra-red radiation, electronic computing machines, etc., than to civil applications, radio broadcasting, television, etc. There is an unusually large number of illustrations."

3648:
Piaget, Jean; Inhelder, Bärbel; Szeminska, Alina. ★**The child's conception of geometry.** Translated from the French by E. A. Lunzer. Basic Books, Inc., New York, 1960. vii+411 pp. $7.50.
Piaget and Inhelder's *The child's conception of space* [Humanities Press, New York, 1956] dealt with topological and projective properties. The present book continues the theme by showing how children go from the more elementary topological concepts to notions of measurement and coordinatization. The most obvious lesson to be learned from these careful and imaginative experiments is that what appears intuitively obvious to the well-trained adult is actually a reflection of many years of experience in which partial insights are attained one step at a time. Required reading for those concerned with mathematical pedagogy at the elementary level, the book contains also material of interest to all teachers and to geometers intrigued by the psychology of their subject.
Kenneth May (Northfield, Minn.)

HISTORY AND BIOGRAPHY

See also 3679.

3649:
Ball, W. W. Rouse. ★**A short account of the history of mathematics.** Dover Publications, Inc., New York, 1960. xxiv+522 pp. $2.00.
Republication of the 4th edition [Macmillan, London, 1908].

3650:
Szabó, Árpád. On the axiomatic foundation of Greek mathematics. Mat. Lapok **10** (1959), 72–121. (Hungarian)
A German version of this paper appeared in Studi Ital. Filologia Classica **30** (1958), 1–51 [MR **21** #2566]. The author discusses the origins of Greek mathematics. In his opinion mathematics became a deductive science under the influence of Eleatic philosophy.
P. Erdös (Birmingham)

3651:
Lauriola, Luca. La scuola matematica italiana. Civiltà delle Macchine **8** (1960), 3–7. (French, German, English and Spanish summaries)
A history from earliest times to the present.

3652:
Subbotin, M. F. Leonhard Euler and the astronomical problems of his time. Voprosy Ist. Estest. i Tehn. **7** (1959), 58–66. (Russian) .

3653:
Verhunov, V. M. N. I. Lobačevskiĭ and mechanics. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1958**, no. 6, 77–89. (Russian)

3654:
Četaev, N. G. Note on classical Hamiltonian theory. Prikl. Mat. Meh. **24** (1960), 33–34 (Russian); translated as J. Appl. Math. Mech. **24**, 40–42.

3655:
Yuškevič, A. P. Blaise Pascal as a scientist. Voprosy Ist. Estest. i Tehn. **7** (1959), 75–85. (Russian)
Originally presented to the section of Friends of French

Science and Culture and to the scientific-technical section of the Union of Soviet Societies of Friendship and Cultural Relations with Foreign Countries, this paper concentrates on Pascal's mathematical work. There are interesting details and references to primary sources. Pascal's embroilment in the struggle between the Jansenists and Jesuits and other non-scientific aspects of his life get only passing though friendly mention.

Kenneth May (Northfield, Minn.)

3656:

Könyves Tóth, Kálmán. F. Bólyai, precursor of modern didactics of mathematics (To the 100th anniversary of his death). Mat. Lapok **10** (1959), 12–22. (Hungarian)

The author discusses some aspects of a little known book of F. Bólyai on "geometry for beginners".

P. Erdös (Birmingham)

3657:

Belousova, V. P.; Dobrovol'skiĭ, V. A.; Il'in, I. G.; Smogorževskiĭ, A. S. Boris Yakovlevič Bukreev. On the hundredth anniversary of his birth. Uspehi Mat. Nauk **14** (1959), no. 5 (89), 181–195. (1 plate) (Russian)

3658:

Belousova, V. P.; Il'in, I. G. Boris Yakovlevič Bukreev (on the centenary of his birth). Ukrain. Mat. Ž. **11** (1959), 312–314. (1 plate) (Russian)

3659:

Nikolaĭ Gur'evič Četaev. Prikl. Mat. Meh. **24** (1960), 3–5 (1 plate) (Russian); translated as J. Appl. Math. Mech. **24**, 1–4. (1 plate)

3660:

A survey of the scientific works of N. G. Četaev. Prikl. Mat. Meh. **24** (1960), 171–200 (Russian); translated as J. Appl. Math. Mech. **24**, 238–287.

3661:

Brockmeyer, E.; Halstrøm, H. L.; Jensen, Arne. The life and works of A. K. Erlang. Acta Polytech. Scandinav. No. 287 (1960), 277 pp.

This is a second, unaltered, edition of the work of which the first edition was in Trans. Danish Acad. Tech. Sci. **1948**, no. 2 [MR **10**, 385].

3662:

Sagoroff, S. Nachruf für Oskar Anderson. Metrika **3** (1960), 89–94.

3663:

Togliatti, Eugenio. Obituary: Francesco Sbrana. Atti Accad. Ligure **15** (1959), 466–471.

3664:

Smirnov, V. I.; Linnik, Yu. V. Obituary: Nikolaĭ Sergeevič Košlyakov. Uspehi Mat. Nauk **14** (1959), no. 3 (87), 115–122. (1 plate) (Russian)

3665:

Kuroš, A. G.; Lyusternik, L. A.; Markuševič, A. I.; Rybkin, G. F. Anisim Fedorovič Bermant. Necrologue. Uspehi Mat. Nauk **14** (1959), no. 5 (89), 117–121. (1 plate) (Russian)

3666:

Mitropol'skiĭ, Yu. A.; Tyablikov, S. V. Nikolaĭ Nikolaevič Bogolyubov. On the fiftieth anniversary of his birth. Uspehi Mat. Nauk **14** (1959), no. 5 (89), 167–180. (1 plate) (Russian)

3667:

Mitropol'skiĭ, Yu. A.; Tyablikov, S. V. Nikolaĭ Nikolaevič Bogolyubov (on the fiftieth anniversary of his birth). Ukrain. Mat. Ž. **11** (1959), 295–311. (1 plate) (Russian)

3668:

Višik, M. I.; Lyusternik, L. A. Sergeĭ L'vovič Sobolev: on his 50th birthday. Uspehi Mat. Nauk **14** (1959), no. 3 (87), 203–214. (1 plate) (Russian)

3669:

Aleksandrov, P. S.; Miščenko, E. F. Lev Semenovič Pontryagin: on his 50th birthday. Uspehi Mat. Nauk **14** (1959), no. 3 (87), 195–202. (1 plate) (Russian)

3670:

Gel'fond, A. O.; Sarmanov, O. V. The eightieth birthday of Sergeĭ Natanovič Bernšteĭn. Izv. Akad. Nauk SSSR. Ser. Mat. **24** (1960), 309–314. (1 plate) (Russian)

A summary of Bernšteĭn's professional life. The list of his publications goes back only to 1950; for a listing of the earlier ones, the authors refer to same Izv. **4** (1940), 249–260; **14** (1950), 193–198 [MR **2**, 114; **11**, 707].

3671:

Volterra, Vito. ★Opere Matematiche: Memorie e note. Vol. IV: 1914–1925. Pubblicate a cura dell'Accademia Nazionale dei Lincei col concorso del Consiglio Nazionale delle Ricerche. Accademia Nazionale dei Lincei, Rome, 1960. 540 pp. Lire 8000.

Twenty-five articles. Earlier volumes are listed in MR **16**, 2; **18**, 268; **19**, 827.

LOGIC AND FOUNDATIONS

See also 3701, 3702, B4129.

3672:

Basson, A. H.; O'Connor, D. J. ★Introduction to symbolic logic. The Free Press, Glencoe, Ill., 1960. vii+175 pp. $3.00.

Third edition of this introductory textbook; the first [University Tutorial Press, Ltd., London, 1953] was listed in MR **17**, 3. Contents: Propositional calculus;

predicate calculus (with emphasis on the notion of satisfiability); and an appendix on the relation between classical syllogism and Boolean algebra.

3673:

Nidditch, P. H. ★Introductory formal logic of mathematics. The Free Press, Glencoe, Ill., 1960. vii + 188 pp. $3.00.

Unchanged (apparently) from the English edition [University Tutorial Press, Ltd., London, 1957; MR **19**, 723].

3674:

★Попов, А. И. [Popov, A. I.] ★Введение в математическую логику. [Introduction to mathematical logic.] Izdat. Leningrad. Univ., Leningrad, 1959. 108 pp. 3.90 r.

From the introduction: "This booklet is intended for those who wish to become acquainted with the foundations of mathematical logic without careful study of various details interesting only to specialists. We have chiefly had in mind philosophers and instructors in logic." The chapter headings are: algebra of logic, mathematical logic and the foundations of mathematics, the law of the excluded middle, the intuitionistic and constructive movements in mathematical logic.

3675:

Black, Max. ★The nature of mathematics: A critical survey. International Library of Psychology, Philosophy and Scientific Method. Littlefield, Adams & Co., Paterson, N.J., 1959. xiv + 219 pp. $1.50.

This is a photographic reprint of a book originally published in 1933. The author had two aims in mind: to present a considered critical exposition of *Principia mathematica*, and to give supplementary accounts of the formalist and intuitionist doctrines in sufficient detail to lighten the paths of all who may be provoked to read the original papers. In particular, the exposition of Brouwer's intuitionism may still prove helpful. In addition, the reprint presents a certain historic interest because the book reflects the state of affairs at a moment when Gödel's results were known but their significance was not yet completely realized. *E. W. Beth* (Amsterdam)

3676:

Łoś, J.; Słomiński, J.; Suszko, R. On extending of models. V. Embedding theorems for relational models. Fund. Math. 48 (1959/60), 113–121.

[For part IV see Łoś and Suszko, Fund. Math. **44** (1957), 52–60; MR **19**, 724.] Following Mostowski, the authors define a homomorphism h of a relational system $\mathfrak{A}=\langle A, R\rangle$ onto a relational system $\mathfrak{B}=\langle B, S\rangle$ as a mapping of A onto B such that $S(h(x), h(y))$ if and only if there exist $u, v \in A$, $h(x)=h(u)$, $h(y)=h(v)$, and $R(u, v)$. A family of homomorphisms h_t, $t \in T$, of $\mathfrak{A}=\langle A, R\rangle$ onto $\mathfrak{A}_t=\langle A_t, R_t\rangle$ is productable if and only if the mapping h, defined by the equation $h(a)(t)=h_t(a)$, is a homomorphism of $\mathfrak{A}$ into $P_{t\in T}\mathfrak{A}_t$. The authors give some rather simple necessary and sufficient conditions for a family of homomorphisms to be productable. Similar results are obtained for families of congruences. Some applications of their results are given. One is a theorem on the embeddability of every finitary relational system into a decidable relational system, and the other is a proof of a result of the reviewer on the embeddability of certain subsystems of a direct product into products over a smaller index set. *C.-C. Chang* (Los Angeles, Calif.)

3677:

Mihăilescu, Eugen. Sur quelques théorèmes de la logique classique. Rev. Math. Pures Appl. 4 (1959), 233–248.

The author adjoins the primitive functor B to his system $L(E, K, R)$ [An. Univ. "C. I. Parhon" Bucureşti. Ser. Acta Logica **1** (1958), no. 1, 173–185; MR **21** #2580] together with the axiom A9: $E\ RKpqq\ Bpq$. The resulting system is demonstrably incomplete, admitting infinitely many free forms equivalent to each other, the shortest of which are the forms Rpp and Bpp. The theorems of the present paper provide normal forms for formulas of the resulting calculus which permit more easily the assignment of truth values. *E. J. Cogan* (Bronxville, N.Y.)

3678:

Shoenfield, J. R. Degrees of formal systems. J. Symb. Logic **23** (1958), 389–392.

In same J. **22** (1957), 161–175 [MR **20** #5130], Feferman showed how to define a consistent axiomatizable theory with an arbitrary recursively enumerable degree of unsolvability. The author now shows that, if the given recursively enumerable degree of unsolvability is not recursive, then the consistent axiomatizable theory can be chosen essentially undecidable. A single non-logical constant is used in defining this essentially undecidable theory. In the course of obtaining this result, the author shows how to define a pair of recursively inseparable recursively enumerable sets with the same degree of unsolvability as an arbitrary non-recursive recursively enumerable set.

In the above-mentioned paper, Feferman gave sufficient conditions for the creativity of a theory; an open question was the creativity of a particular theory of Kreisel. The author gives more general conditions for creativity and he applies them to establish the creativity of Kreisel's theory. *G. F. Rose* (Pacific Palisades, Calif.)

3679:

Hajnal, András. The work of J. von Neumann in the axiomatic set-theory. Mat. Lapok **10** (1959), 5–11. (Hungarian)

3680:

Kreisel, Georg; Lacombe, Daniel. Ensembles récursivement mesurables et ensembles récursivement ouverts ou fermés. C. R. Acad. Sci. Paris **245** (1957), 1106–1109.

This paper is a continuation of an earlier paper of Lacombe's [same C. R. **245** (1957), 1040–1043; **246** (1958), 28–31; MR **21** #5572]; terminology is the same as in that paper. The authors' principal result can be motivated as follows: Of the two classical proofs of the uncountability of [0, 1], the diagonal argument is 'recursively correct', i.e., the set of all (indices of) recursive real numbers belonging to [0, 1] is productive [Dekker, Trans. Amer.

Math. Soc. **78** (1955), 129–149; MR **16**, 663; especially pp. 129–130]; while the other argument, from the fact that countable sets have (Lebesgue) measure 0 while [0, 1] has measure 1, is not recursively correct, because in any reasonable sense the set of all recursive real numbers in [0, 1] has 'recursive measure' 0. More precisely (theorem VI), the set of all such numbers can be covered by a recursively open set of arbitrarily small measure. (On the other hand (theorem V), it cannot be covered by a recursively open set whose measure is a recursive real number, unless that measure is $\geqq 1$.)

J. Myhill (Stanford, Calif.)

3681:

Spector, Clifford. Measure-theoretic construction of incomparable hyperdegrees. J. Symb. Logic **23** (1958), 280–288.

It is widely known (but never stated in print) that the Kleene-Post theorem on incomparable degrees [Kleene and Post, Ann. of Math. (2) **59** (1954), 379–407; MR **15**, 772] is an immediate corollary of the fact that the square of a Hausdorff space without isolated points is not the union of a countable number of continuous curves, i.e., graphs of (partial) functions $f(x)=y$ or $f(y)=x$ continuous in the topology induced by their domains. For every partial recursive operator is continuous in Baire space. A similar argument for incomparable hyperdegrees is not possible, because hyperarithmetic operators are in general discontinuous. However the square of Baire space is easily seen not to be the union of countably many (Lebesgue, plane) measurable curves, and every hyperarithmetic operator surely possesses an (analytic and consequently) measurable graph. This establishes the existence of incomparable hyperdegrees (Theorem 2). Further, more sophisticated results of the same kind are obtained; e.g., a natural definition of the 'hyperjump' operation $\mathbf{a}'$ is given, in analogy with the Kleene-Post jump operation [loc. cit., p. 384]; and it is proved that there exist incomparable hyperdegrees satisfying $\mathbf{a}\cup\mathbf{b}=\mathbf{a}'\cup\mathbf{b}'$.

J. Myhill (Stanford, Calif.)

3682:

Friedberg, Richard M.; Rogers, Hartley, Jr. Reducibility and completeness for sets of integers. Z. Math. Logik Grundlagen Math. **5** (1959), 117–125.

The main result of this paper is a generalization of the principal theorem of the reviewer's paper in same Z. **1** (1955), 97–108 [MR **17**, 118], which can be stated as follows: An r.e. (recursively enumerable) set is creative if and only if it is (many-one) complete. To facilitate exposition, we change the authors' notation and terminology slightly to accord with the reviewer's. For every effective operation ϕ_0: $F\times F\to F$ define a canonical extension $\phi: F\times V\to V$ by setting $\phi\omega_n\beta=\sum_{\omega_k\subset\beta}\phi_0\omega_n\omega_k$, and call ϕ a 'normal map' if it can be obtained in this way. Every such ϕ induces a reducibility relation R_ϕ at least on r.e. sets, as follows: $\alpha R_\phi\beta$ if and only if $\alpha'=f^{-1}\phi\beta\beta'$ with a recursive f. $\alpha\in F$ is called 'ϕ-complete' if $\beta R_\phi\alpha$ for every $\beta\in F$, 'ϕ-creative' if given $\gamma\in F$ we can effectively find an element of $(\phi\alpha\alpha'-\phi\alpha\gamma)+(\phi\alpha\gamma-\phi\alpha\alpha')$. The principal result of the paper under review is that for normal ϕ, ϕ-completeness and ϕ-creativity coincide. With $\phi\alpha\beta=\beta$, this specializes to the reviewer's result; suitable choices of ϕ yield analogous results for truth-table, bounded truth-table and Turing reducibilities. In later sections, the authors relate these reducibilities to the jump operation introduced by Kleene and Post [Ann. of Math. (2) **59** (1954), 379–407; MR **15**, 772; p. 384], and consider various new reducibilities, some of them not induced by normal maps.

J. Myhill (Stanford, Calif.)

3683:

Tugué, Tosiyuki. On predicates expressible in the 1-function quantifier forms in Kleene hierarchy with free variables of type 2. Proc. Japan Acad. **36** (1960), 10–14.

The results contained in this paper appear in greater detail and with some improvements in the author's paper in Comment. Math. Univ. St. Paul. **8** (1960), 97–117 [MR **22** #1515].

E. J. Cogan (Bronxville, N.Y.)

3684a:

Hu, Shih-hua. Recursive algorithms. Theory of recursive algorithms I. Acta Math. Sinica **10** (1960), 66–88. (Chinese. English summary)

3684b:

Hu, Shih-hua; Loh, Chung-wan. Kernel functions. Theory of recursive algorithms II. Acta Math. Sinica **10** (1960), 89–97. (Chinese. English summary)

3684c:

Hu, Shih-hua. Normal forms of recursive functions. Theory of recursive algorithms III. Acta Math. Sinica **10** (1960), 98–103. (Chinese. English summary)

In this sequence of papers the author (with a co-author in paper II) presents a theory of recursive algorithms. He begins with a discussion of the various known notions of recursiveness and computability. He then points out that, although all notions have been proved to be equivalent, each notion suffers from some 'defect'. Next he states: "We can construct a theory of recursive algorithms (which will be equivalent to each of the above-mentioned theories) which possesses the good points and which either does not have or minimizes the bad points of each of the above theories. We hope that the theory presented in this paper satisfies those requirements." Since what he seeks will in the end be proved equivalent to each of the known notions, it is hard for the reviewer to see precisely what is gained by such an exposition. However, this series of articles may be a basis for the training of future logicians in China who are interested in the foundation of recursive function theory.

The contents of the papers are as follows.

In I, the author develops the notions of recursive algorithms and recursive functions defined, or partially defined, on a free semi-group freely generated by a finite alphabet. The ideas behind his definition of an algorithm seem to follow very closely the one given by Kleene in his book. The author then proves, in a sketchy manner, the equivalence of his notion of recursiveness to the notion of normal algorithms of Markov and to the notion of machine computability of Turing.

In II, the authors define a narrower class of recursive functions called the kernel functions. The class of kernel functions corresponds in a natural manner to the class of primitive recursive functions. They prove that each kernel

function can be represented in a 'normal form'. Using this representation the authors prove that there exist universal normal algorithms (in the sense of Markov) and universal Turing machines.

In III, the author discusses various representation theorems for recursive functions due to several authors (for instance, Kleene's representation of a general recursive function in terms of a fixed primitive recursive function and the μ operator). He obtains similar results by showing that every recursive function (in his sense) can be represented in terms of a fixed kernel function and some operators. *C.-C. Chang* (Los Angeles, Calif.)

3685:

Grzegorczyk, Andrzej. Some approaches to constructive analysis. Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting), pp. 43–61. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. viii+297 pp. $8.00.

This paper contains improvements of some fundamental results of (classical) analysis which are suggested by intuitionistic mathematics. If $\mathfrak{A}$ denotes the (prenex) formula

$$(\alpha_1)(E\beta_1)\cdots(\alpha_n)(E\beta_n)(x)R(\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n, x)$$

with R recursive in the α's and β's (and α, β ranging over N^N), let $\mathfrak{A}_M$ denote the relativization of $\mathfrak{A}$ in which the α and β range over M. Let Γ be a set of operations from functions to functions. Generally speaking, an improvement of $\mathfrak{A}$ consists in showing (i) that $\mathfrak{A}_M$ is true in every class M closed under the operations Γ and not only in the class M_0 of all functions, and (ii) that there are operations $\lambda\alpha_1 B(\alpha_1), \cdots, \lambda\alpha_1, \cdots, \alpha_n B(\alpha_1, \cdots, \alpha_n)$ of Γ such that

$$(\alpha_1)\cdots(\alpha_n)(x)R(\alpha_1, \cdots, \alpha_n, B_1, \cdots, B_n, x).$$

Examples: If $\mathfrak{B}$ states (in the natural way) that every real number a with a (Cauchy) representation α, i.e.,

$$(n)[|\alpha(n)/(n+1)-a| < 1/(n+1)],$$

has a (Dedekind) representation β, i.e.,

$$(p)(q)[p/q > a \equiv \beta(p, q) = 0],$$

then $\mathfrak{B}$(i) holds with Γ the class of recursive operations Γ_K. It is to be remarked that the use of a free variable a over real numbers is inessential: replace '$|\alpha(n)/(n+1)-a| < 1/(n+1)$' by

$$(Er)(m)[|\alpha(n)/(n+1)-\alpha(m)/(m+1)| < 1/(n+1)+1/(m+1)-1/r]$$

and '$(p)(q)[p/q > a \equiv \beta(p, q) = 0]$' by

$$(p)(q)[(Em)\{p/q \geqq [\alpha(m)+1]/(m+1)\} \equiv \beta(p, q) = 0].$$

Similar results for different representations of continuous functions of a real variable. The obvious fact that $\mathfrak{B}$(ii) does not hold for Γ_K is sharpened: Following Mazur the author shows that there are no Banach-Mazur functionals $B_1, \cdots, B_n$, described in detail for the first time in this paper, which satisfy $\mathfrak{B}$(ii) even if the α are only required to range over recursive functions. This follows from the continuity of B-M functionals at recursive arguments. Following Mostowski [Fund. Math. **44** (1957), 37–51; MR **19**, 934] the failure of $\mathfrak{B}$(ii) can be derived from the failure of $\mathfrak{B}_M$ when the α and β range over recursive sequences of functions; i.e., the uniformity required by $\mathfrak{B}$(ii) for all α cannot be satisfied even if it is only required for each recursive sequence of α.—The author observes that if $\mathfrak{A}$(ii) is proved classically for Γ_K it can usually also be proved intuitionistically. This has been rigorously established for certain fragments of classical analysis by the reviewer [J. Symb. Logic **23** (1958), 155–182 in the following precise sense: if recursion equations for $B_1, \cdots, B_n$ of Γ_K are explicitly given, $\mathfrak{A}$(ii) is equivalent to the functional character of the recursion equations (which is expressed by $(\alpha)(Ez)F(\alpha, z)$, F recursive) together with a purely universal formula; but such formulae are provable intuitionistically if they are proved in elementary fragments of classical analysis. *G. Kreisel* (Reading)

3686:

Kleene, S. C. Countable functionals. Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting), pp. 81–100. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. viii+297 pp. $8.00.

Natural numbers are objects of type 0, and functions whose arguments are type $j-1$ objects and whose values are natural numbers are type j objects. For $j \geqq 2$, type j objects are called functionals. A type 2 functional is said to be countable if its value for any (type 1) argument depends on only a finite number of values of that argument. For every countable type 2 functional F, there is a type 1 function f such that for every argument g of F there is an argument x of g such that $F(g)+1=f(x)$; f is called an associate of F. The definition extends to type j objects for $j > 2$ which also have type 1 associates. The relation between countable functionals and their associates is examined, and conditions for constructibility of the functionals are obtained. A functional is said to be recursively countable if it is countable and has a general recursive associate. The author investigates the properties of this class and of general recursive functionals whose arguments are restricted to be countable. The paper closes with an investigation of the applicability of the results to finite types of the following more general character. Natural numbers are of type 0; an n-place function with arguments of types $t_1, \cdots, t_n$ and value type s has type $(s; t_1, \cdots, t_n)$. The following classes of types are distinguished: (1) the pure types: $0, (0; 1), \cdots, (0; n)$; (2) the special types: $0, (0; t_1, \cdots, t_n)$ where $t_1, \cdots, t_n$ are pure types; (3) the numerical types in which $s=0$. *E. J. Cogan* (Bronxville, N.Y.)

3687:

Lacombe, Daniel. Quelques procédés de définition en topologie recursive. Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting), pp. 129–158. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. viii+297 pp. $8.00.

The paper begins with a summary of recursive (rec.) analysis of metric spaces. The examples include intuitionistically valid results such as (i) Picard's theorem on differential equations which hold even for the informal notion of effective function (i.e., independently of the identification: effective=rec.) and (ii) Dini's theorem if

interpreted as: if α^* and α_n are rec., $(n)(\alpha_n \leqq \alpha_{n+1})$, and if there exists some (effective) function β which satisfies $(m)(\alpha^* - m^{-1} \leqq \alpha_{\beta(m)} \leqq \alpha^*)$, then α_n rec. converges to α^*. —The bulk of the paper concerns spaces $\mathbf{X}$ with a numbering ν of a denumerable base $\mathscr{X}$ of neighbourhoods $X_{\nu(X)}$ of $\mathbf{X}$. The space of continuous mappings from such an $\mathbf{X}$ to such a $\mathbf{Y}$ is denoted by $\mathbf{X} \Rightarrow \mathbf{Y}$, where (in the compact-open topology) each basic neighbourhood of the function space is the intersection of a finite number of sets $\{f | f \in \mathbf{X} \Rightarrow \mathbf{Y}, f(\bar{X}_{n_i}) \subset Y_{m_i}\}$, say $(\cdots\langle n_i, m_i\rangle\cdots)$. From effective numberings of bases for $\mathbf{X}$ and $\mathbf{Y}$, one gets one for $\mathbf{X} \Rightarrow \mathbf{Y}$, if one can decide, for any finite set of pairs $\langle n_i, m_i\rangle$, whether $(\cdots\langle n_i, m_i\rangle\cdots)$ is empty. A method due to J. P. Serre reduces this decision, for a general class of spaces, to questions of the form: is $\bigcap X_{n_i}$, $\bigcap Y_{m_i}$ empty? For spaces with numberings as described, the notions of (rec.) strong and weak approximations are introduced: a subset $\mathfrak{A}$ of $\mathscr{X}$ is (called) an approximation in $\mathbf{X}$ if it satisfies obvious closure conditions, $\mathfrak{A}$ is a strong one (s.a.) if, for some $x \in \mathbf{X}$, $\mathfrak{A} = \{X | x \in X \in \mathscr{X}\}$, weak (w.a.) if a unique x satisfies $(X)(X \in \mathfrak{A} \rightarrow x \in X)$; in particular, if x has a rec. s.a., i.e., is a strong limit, one can decide, for each $X \in \mathscr{X}$, whether $x \in X$. Any approximation $\mathfrak{F}$ in $\mathbf{X} \Rightarrow \mathbf{Y}$ associates, in a natural way, with each $\mathfrak{A}_X$ in $\mathbf{X}$ an $\mathfrak{A}_Y$ in $\mathbf{Y}$. An element $x \in \mathbf{X}$ which is the (strong) limit of $\mathfrak{A}_X$ is called (p. 149) a point of convergence of $\mathfrak{F}$ if and only if the associated $\mathfrak{A}_Y$ is s.a. in $\mathbf{Y}$. The author states elegantly a number of results for functionals of lowest type in terms of the notions just described. While, as seen below, the restriction to s.a. in the definition of point of convergence (p. 149) is counter-intuitive, most of the examples are valid also on a good definition.—It is to be emphasized that at higher types the author's notion of continuity does not coincide with that induced by Kleene's countable (COU) functionals [#3686] or, except for special choices of $\mathfrak{F}_i$, with the reviewer's continuous (CON) functionals of $C(\mathfrak{F}_1, \cdots, \mathfrak{F}_n)$ [same Proc., pp. 101–128; MR **21** #5568] where continuity is defined with respect to the 'proper' topology of p. 127 (which is, essentially, the so-called smallest topology associated with $C(\mathfrak{F}_1, \cdots, \mathfrak{F}_n, \mathfrak{F}_{n+1})$ functionals). For, if N_0 denotes the set of natural numbers, N_{i+1} is $N_i \Rightarrow N_0$, α, β range over N_1, φ over N_2, then the modulus of continuity Φ_M of φ at the null function, i.e., $\mu_y(\alpha)(\beta)\{(x)[x \leqq y \rightarrow \alpha(x) = \beta(x) = 0] \rightarrow \varphi(\alpha) = \varphi(\beta)\}$ is continuous on the author's topology, in fact, has a totally convergent, rec. s.a., but is not COU (and hence not rec. COU), nor in $C(\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3)$ of (CON) unless, e.g., $\mathfrak{F}_2$ is restricted to maximal representing functions $\mathfrak{F}_2{}^M$, loc. cit. p. 115 [=principal associated function of (COU), p. 83]. Both (COU) and (CON) (pp. 83 and 115) explicitly state that they do not restrict themselves to the latter; in fact, under such a restriction, the most useful result yet obtained in this area would be false, namely theorem 3 of (COU) or the slightly more general 4.141 of (CON): if φ and ψ are COU and $A(\varphi, \psi)$ is rec. COU, for some rec. COU Ψ, $A[\varphi, \Psi(\varphi)]$. If φ, ψ range over $C(\mathfrak{F}_1, \mathfrak{F}_2{}^M, \cdots, \mathfrak{F}_n{}^M)$, there is still a Ψ by the appendix of (CON), since we have rec. maximal representing functions for the finite functionals and so the density theorem applies. But Ψ need not have a rec. maximal representing function. Intuitively, the situation is quite clear: one operates not on abstract objects (functionals of higher type) but on their spatio-temporal representations by representing functions (associated ones, approximations). So the primary notion is that of an operation, of lowest type, on the latter. The operation may be quite effective if applied only to maximal representations, but not in general. In terms of the primary notion, classes of functionals can be defined [(COU) 1.1–1.4 on pp. 82–84; (CON) 4.122 on p. 115], and hence a topology on the space of type i functionals in terms of the given class of $(i+1)$ functionals: while this is the logical order, for some mathematical purposes it is of course useful to relate this to topologies introduced in the reverse direction.—The author discusses briefly, but interestingly, the possible significance of rec. analysis, and concludes that it may some day lead to serious mathematical problems. It may fairly be said that the change in analysis over the last fifty years, with its move from the study of non-denumerable sets of point-functionals to denumerable bases of neighbourhoods, was a pre-requisite for rec. analysis; and so many of the most interesting purely mathematical problems in this connection may already be solved, particularly those related to continuity rather than constructivity proper. But even in this case, a few convenient definitions of rec. analysis would have a value as a language in terms of which certain informally recognizable differences between different proofs can be stated precisely.—The results on Banach-Mazur functionals quoted by the author have since been published by Grzegorczyk [#3685] and Friedberg [Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 1–5; MR **20** #3071]. It is to be remarked that the questions on $\int_0^1 f dg$, g of variation r, on pp. 157–158, were intended by the author as exercises and not as open problems. The answers are positive and, in fact, $J(f, g, r)$ is even rec. COU and not only rec. s.a. *G. Kreisel* (Reading)

3688:

Mostowski, Andrzej. On various degrees of constructivism. Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting), pp. 178–194. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. viii+297 pp. $8.00.

The author denotes by $\Pi_\sigma{}^0 \cap \Sigma_\sigma{}^0$ (or K_σ) the class of sets of natural numbers which are recursive in hyperarithmetic sets of level $<\sigma$, and by $\Pi_n{}^1$ $[\Sigma_n{}^1]$ those definable by means of n function quantifiers (over N^N) starting with a universal [existential] one. He denotes by $\mathbf{K}_\sigma$ the class of real numbers a such that, for some function α whose graph is in K_σ, $(n)[|\alpha(n)/(n+1) - a| < (n+1)^{-1}]$. He observes that every a in $\mathbf{K}_\sigma$ has a Dedekind representation in K_σ if and only if σ is a limit number. He raises the question of comparing the hierarchy of K_σ with the ramified hierarchy of types which has since been done by Kleene for $\sigma < \omega_1$ [Compositio Math. **14** (1958), 23–40; MR **21** #2586]. A very simple argument yields the following useful result: the comprehension principle for number-theoretic functions (or, equivalently, the principle of the least upper bound) does not hold in $K(\mathbf{K})$ if K can be enumerated by a function which is definable by means of a second order formula whose (second order) quantifiers range over K. This applies to the classes $\Pi_1{}^0 \cap \Sigma_1{}^0$, K_ω, $\Pi_1{}^1 \cap \Sigma_1{}^1$, $\Pi_2{}^1 \cap \Sigma_2{}^1$ (by a later communication of the author) and in fact to K_σ, $\sigma < \omega_1$ for limit numbers σ, by Kleene [loc. cit.]. However, the principle does apply to $\bigcup \Pi_n{}^1$ provided the quantifiers are limited to constructible sets in Gödel's sense.—Among the many incidental

observations we have: If Bernays' theory of classes without the axiom of infinity is adapted to second order arithmetic, $(EX)(n)[n \in X \equiv G(n)]$, where G is a second order formula, is provable only if $G(n)$ is provably equivalent to one of a finite set of first order formulae. Among the questions raised is one solved by Shoenfield [Proc. Amer. Math. Soc. **9** (1958), 690–692; MR **20** #2281] who showed that $\hat{\phi}\ (\phi \in K_0)$ is not in $\Pi_3{}^0[\phi]$. It is to be noted that K_0 can of course be enumerated by a function of $\Pi_3{}^0$.—The author emphasizes that his results have a somewhat disjointed character: this seems unavoidable because he avowedly tries to do without the intuitionistic primitive notions, thereby depriving himself of just those concepts which are needed for a systematic approach to constructivity. *G. Kreisel* (Reading)

3689:

Dekker, J. C. E.; Myhill, J. The divisibility of isols by powers of primes. Math. Z. **73** (1960), 127–133.

Let ε be the set $\{0, 1, 2, \cdots\}$ of natural numbers and $\bar{\varepsilon}$ be the set consisting of the natural numbers and the symbol ∞. Every isol X determines a unique sequence $\{v_n\}$ of elements of $\bar{\varepsilon}$ such that (1) if v_n is in ε then $p_n{}^{v_n}|X$ and not $p_n{}^{v_n+1}|X$, and (2) if $v_n = \infty$, then $p_n{}^k|X$ for all k. The sequence $\{v_n\}$ is called the characteristic of X. Two questions arise: (1) Does there exist for each sequence $\{v_n\}$ of elements of $\bar{\varepsilon}$ an isol X having $\{v_n\}$ as its characteristic? (2) Does there exist for each sequence $\{V_n\}$ of isols an isol X such that, for all n, $p_n{}^{V_n}|X$ and not $p_n{}^{V_n+1}|X$? The first theorem answers the second question affirmatively, thus providing an affirmative answer for the first question as well. Theorem 2 proves an analogous result in the case that $\{V_n\}$ is an r.e. sequence of cosimple isols and X is a cosimple isol. An isol is cosimple if it is representable by a set that is either finite or the complement of a simple set. For notation and basic notions see Dekker, Math. Z. **70** (1958), 113–124, 250–262 [MR **20** #5133, 5134]. *E. J. Cogan* (Bronxville, N.Y.)

3690:

Robinson, A. Solution of a problem of Tarski. Fund. Math. **47** (1959), 179–204.

The problem of A. Tarski solved in this paper is the one raised in Note 21, p. 57 of *A decision method for elementary algebra and geometry*, 2nd ed. [Univ. Calif. Press, Berkeley, Calif., 1951; MR **13**, 423]; namely, to provide a decision procedure for those elementary sentences (of an applied first-order predicate calculus), expressed in terms of equality, order, addition, multiplication and a unary predicate $A(x)$, which are true in the field of real numbers when $A(x)$ is satisfied only by the algebraic real numbers. In addition the corresponding problem for the field of complex numbers is solved.

A theory K^{**} is constructed whose sentences have equality, order, addition, multiplication and $A(x)$ as their only primitive predicates and functions, and which has the real numbers as a model when $A(x)$ is satisfied only by the algebraic real numbers. The theory K^{**} is then shown to be complete in the sense that for any sentence X of K^{**} either X or $\sim X$ is a theorem of K^{**}. Hence one can conclude that a sentence of K^{**} is a theorem of K^{**} if and only if it is true in the model of the real numbers. One can then also conclude that to decide whether or not a sentence X of K^{**} is true in the model of the real numbers one need only enumerate the theorems of K^{**} and examine the sequence for either X or $\sim X$.

The proof that K^{**} is complete depends upon a theorem asserting that a theory K^*, closely related to K^{**}, is model-complete. The proof of this latter theorem is long (almost half the paper) and ingenious. *P. C. Gilmore* (Yorktown Heights, N.Y.)

3691:

Robinson, Julia. The undecidability of algebraic rings and fields. Proc. Amer. Math. Soc. **10** (1959), 950–957.

Let **R** be the ring of the algebraic integers of a field **F** of finite degree over the rationals. It is shown that the set **N** of natural numbers is arithmetically definable in **R**. A routine transformation is given leading from a statement of the arithmetic of **N** into an equivalent statement of the arithmetic of **R**. Hence **R** is undecidable. Further, it is shown that **R** is arithmetically definable in **F**, and hence **N** is arithmetically definable in **F** and **F** is undecidable. The proof utilizes the fact that every statement of the arithmetic of **F** can be transformed into an equivalent statement of the arithmetic of **R** by (i) replacing each variable whose range is **F** by the ratio of two variables with range **R**, (ii) adding the condition that the denominator is not 0, and (iii) clearing the resulting equation of fractions. *R. M. Martin* (Bonn)

SET THEORY

See also 3676.

3692a:

Choquet, Gustave. Ensembles $\mathscr{K}$-analytiques et $\mathscr{K}$-sousliniens. Cas général et cas métrique. Ann. Inst. Fourier. Grenoble **9** (1959), 75–81.

3692b:

Choquet, Gustave. Forme abstraite du théorème de capacitabilité. Ann. Inst. Fourier. Grenoble **9** (1959), 83–89.

3692c:

Choquet, Gustave. Sur les points d'effilement d'un ensemble. Application à l'étude de la capacité. Ann. Inst. Fourier. Grenoble **9** (1959), 91–101.

3692d:

Choquet, Gustave. Sur les G_δ de capacité nulle. Ann. Inst. Fourier. Grenoble **9** (1959), 103–109.

Les quatre articles se rattachent aux résultats antérieures de l'A. [C. R. Acad. Sci. Paris **243** (1956), 635–638; mêmes Ann. **5** (1953/54), 135–295; MR **18**, 295].

a. Pour des raisons particulières (la théorie des capacités!) l'A. définit les ensembles $\mathscr{K}$-analytiques et $\mathscr{K}$-sousliniens d'une façon différente de la façon usuelle. Un espace topologique A est dit $\mathscr{K}$-analytique [et classique] s'il est séparé et s'il est l'image continu d'un $\mathscr{K}_{\sigma\delta}$ d'un espace bicompact [de l'espace des nombres irrationnels] ($\mathscr{K}$ est la classe des ensembles compacts). Un ensemble A d'un espace topologique séparé E est dit $\mathscr{K}$-souslinien dans E si A est le résultat d'une opération de Souslin portant sur des compacts de E. Pour qu'un $X, X \subset E$

(E étant séparé), soit $\mathscr{K}$-souslinien, il faut et il suffit que X soit $\mathscr{K}$-analytique et contenu dans un $\mathscr{K}_\sigma$ de E (th. 1). Un espace métrisable est analytique classique si et seulement s'il est $\mathscr{K}$-analytique (th. 2). On ne connaît aucun espace $\mathscr{K}$-analytique non-plongeable dans un $\mathscr{K}_\sigma$.

b. Pour un ensemble E soit $\mathscr{H}$ une partie de $\mathfrak{P}E$ vérifiant $\emptyset \notin \mathscr{H}$, $\mathscr{H}_\delta \subset \mathscr{H}$, $\mathscr{H}_\sigma \subset \mathscr{H}$; on définit la capacité abstraite sur $(E, \mathscr{H})$: chaque fonction croissante de $\mathfrak{P}E$ sur $\bar{R}=R[-\infty, \infty]$ telle que pour toute suite décroissante $H_n \in \mathscr{H}$ on ait $f(\bigcap H_n)=\lim fH_n$ et pour toute suite croissante $X_n \in \mathscr{H}$ on ait $f(\bigcup X_n)=\lim fX_n$. Un X, où $X \subset E$, est dit $(f, \mathscr{H})$-capacitable si $fX=\sup fY$ $(Y \in \mathscr{H} \cap \mathfrak{P}X)$. Théorème 1: f étant une capacité abstraite sur $(E, \mathscr{H})$, chaque ensemble $\mathscr{H}$-souslinien est $(f, \mathscr{H})$-capacitable. L'A. indique quelques généralisations de ce théorème.

c. Pour la terminologie voir Brelot [Bull. Sci. Math. (2) **65** (1941), 72–98; MR **7**, 15]. L'A. démontre que pour chaque $X \subset E$ et chaque nombre $\varepsilon > 0$ il existe un ensemble ouvert $G \subset E$ vérifiant $e(X) \subset G$ et $\operatorname{cap}^* (X \cap G) < \varepsilon$ (th. 1). (eX désigne l'ensemble des points de E en lesquels X est effilé.) Il existe une partition de X en deux ensembles X_1, X_2 tels que $\operatorname{cap}^* \bar{X}_1 \leqq \operatorname{cap}^* X$, $\operatorname{cap}^* X_2 < \varepsilon$ (th. 2). Autrement dit, la capacité extérieure est stabilisable dans le sens que voici. E étant un espace topologique et $f: \mathfrak{P}E \to [0, \infty]$ une fonction croissante, f est dite stabilisable si pour chaque $X \subset E$ et chaque $\varepsilon > 0$ il existe une partition de X en X_1, X_2 vérifiant $f\bar{X}_1 \leqq fX$, $fX_2 < \varepsilon$; X est stable pour f si $fX=f\bar{X}$.

Le théorème 1 montre comment E est rare au voisinage des points de eX; la relation $\operatorname{cap}^* E=0$ équivaut à ce que E soit effilé en chacun de ses points. E étant topologique, soit f une fonction stabilisable de $\mathfrak{P}E$ dans $[0, \infty]$; si f est dénombrablement sous-additive, alors pour chaque $X \subset E$ et $\varepsilon > 0$ il y a une partition de X en X_1, X_2 tels que X_1 est stable et $fX_2 < \varepsilon$ (th. 3). L'A. donne d'autres précisions concernant les capacités alternées et les noyaux auxquels s'appliquent les théorèmes 1, 2.

d. Soit $f: E \to \bar{R}$ une fonction semi-continue inférieurement; l'ensemble I_f des points x vérifiant $fx=\infty$ est un G_δ; si en particulier $E=R^p$ $(p \geqq 3)$ et $f=N_\mu$ pour une mesure $\mu \geqq 0$ et un noyau $N \geqq 0$ semi-continu inférieurement, alors $\operatorname{cap}^* I_f=0$; Deny a prouvé le réciproque [C. R. Acad. Sci. Paris **224** (1947), 524–525; MR **8**, 380]; dans la note présente l'A. précise ce résultat en prouvant le théorème que voici: Si A est un ensemble de R^p $(p \geqq 3)$ qui est un G_δ non vide de capacité extérieure nulle, il existe une mesure μ portée par A, dont le potentiel N_μ est infini en A et fini hors de A. (Dans le cas de Deny la mesure μ n'était pas portée par A.) *Đ. Kurepa* (Zagreb)

3693:

Novotný, Miroslav. Über quasi-geordnete Mengen. Czechoslovak Math. J. **9** (84) (1959), 327–333. (Russian summary)

For any ordinal number ν let $\aleph_\nu \oplus \omega_\nu{}^* = \beta_\nu$ denote the order type consisting of an antichain of power $\aleph_\nu$ followed by a chain of type $\omega_\nu{}^*$. Let ${}^{\beta_\nu}I2$ denote the set of all functions from β_ν to $I2=\{0, 1\}$ quasi-ordered in the following way: if $f, g \in {}^{\beta_\nu}I2$, then $f \leqq g$ means that for every $x \in \beta_\nu$ satisfying $fx > gx$ there exists some $x' > x$, $x' \in \beta_\nu$, such that $fx' < gx'$ [cf. G. Birkhoff, *Lattice theory*, Amer. Math. Soc., New York, 1948; MR **10**, 673; in particular p. 9]. The author proves the $\aleph_\nu$-universality of the quasi-ordered set ${}^{\beta_\nu}I2$ in the sense that this quasi-ordered set contains isomorphically every quasi-ordered set of power $\leqq \aleph_\nu$ (theorem 1). Another kind of quasi-ordered relation is exhibited among sequences by the relation "is a subsequence of". In particular, let $I\omega_\nu{}^{I\omega_{\nu+2}} = P_\nu$ mean the set of all $\omega_{\nu+2}$-sequences of ordinals $< \omega_\nu$ structured by the relation "is a subsequence of"; then again P_ν is quasi-ordered and every quasi-ordered set of cardinality $\leqq \aleph_\nu$ is isomorphic to a subset of P_ν (theorem 2). *Đ. Kurepa* (Zagreb)

3694:

Bagemihl, Frederick. Planar sections of spatial sets. Math. Z. **72** (1959/60), 362–366.

Strengthening a theorem of Sierpiński [Fund. Math. **38** (1951), 1–13; MR **14**, 26], the author proves the following: Let (1) $R^3=E_1 \cup E_2 \cup E_3$; if every x-plane intersects E_1 in a finite set and if every y-plane intersects E_2 in $<c$ points, then every z-plane with the exception of $\leqq \aleph_0$ cases intersects E_3 in a planar set which is everywhere of power c, of category 2 and of positive exterior measure. The author considers several propositions concerning partitions (1) and corresponding sections with coordinate planes, and in such terms finds four new propositions, each equivalent to the continuum hypothesis (H). E.g., (th. 5) $\mathrm{H} \Leftrightarrow \mathfrak{K} \wedge \mathfrak{U}_K$, where $\mathfrak{U}_K$ means that there exists a decomposition of the form (1), a set P_1 of second category of x-planes, a set P_2 of second category of y-planes and a non-countable set P_3 of z-planes such that $k(E_1 \cap S_1) \leqq \aleph_0$, $k(E_2 \cap S_2) < c$, $E_3 \cap S_3 \in \mathrm{I}$, where $S_i \in P_i$ $(i=1, 2, 3)$, and where $\mathfrak{K}$ means that the union of $<c$ sets $\in \mathrm{I}$ is again of the first category. Terminology: $P_1 \in \mathrm{II}$ means that P_1 is of second category, i.e. (because P_1 is composed of x-planes), the union of all the members of P_1 meets the x-axis in a set of category II [cf. also F. Bagemihl, Bull. Amer. Math. Soc. **65** (1959), 84–88; MR **21** #3337].

Đ. Kurepa (Zagreb)

COMBINATORIAL ANALYSIS

3695:

★Proceedings of Symposia in Applied Mathematics. Vol. X: Combinatorial analysis. American Mathematical Society, Providence, R.I., 1960. vi+311 pp. $7.70.

The 24 papers will be reviewed individually.

3696:

Harary, Frank. On the measurement of structural balance. Behavioral Sci. **4** (1959), 316–323.

The degree of balance of a signed linear directed graph S can be defined in many ways, in particular as $\beta(S)= C^+(S)/C(S)$, where $C^+(S)$ is the number of positive cycles and $C(S)$ the total number of cycles. A number of theorems are established related to $\beta(S)$ of unbalanced graphs. In particular if S contains n blocks (substructures without cut points) $\beta=\sum C_i^+/\sum C_i$. Further, a necessary and sufficient condition for $\beta=0$ is that S be a Husimi structure consisting of at least one cycle. An upper bound is established on the minimum β and a lower bound on the

maximum of β over all block structures of index m, as a function of m, where $m=q-p+1$, q is the number of lines and p the number of points of a connected graph; and the conjecture is offered that the indicated minimum or maximum can be achieved for an unbalanced block structure of any index m.

Another definition of degree of balance is in terms of a negation-minimal set of lines, i.e., the minimum number of lines whose signs must be changed to achieve balance. Similarly deletion-minimal sets are defined. It is shown that the two sets have the same cardinal number.

The sociological context in which this theory is pursued is the postulate (for which some empirical corroboration exists) that social groups tend to increase their degree of balance in that changes tend to occur in such a way that the corresponding signed linear graph becomes more balanced in some sense. These changes can occur as changes in sign in the binary relations among the members (liking, disliking, etc.), as deletion or addition of such relations, or as expulsion or addition of members.

A. Rapoport (Ann Arbor, Mich.)

ORDER, LATTICES

3697:

Hammer, Preston C. Syntonicity of functions and the variation functional. Rend. Circ. Mat. Palermo (2) 8 (1959), 145–151.

Let f be a real-valued function defined on a simply ordered set S; $P: x_1 \leqq \cdots \leqq x_k$ be a partition of S, $k \geqq 2$; $V(P,f)=\sum_1^{k-1} |f(x_{i+1})-f(x_i)|$ be the variation of f on P and $V(f)=\sup_P V(P,f)$ the total variation of f on S. The variation functional $V(f)$ has a subadditivity property $V(f+g) \leqq V(f)+V(g)$. In this paper are studied pairs of functions h, l of bounded total variation, for which $V(h+l)=V(h)+V(l)$. The basic concept of syntonic functions is defined as follows. Functions f, g on S are called syntonic, if for $\varepsilon>0$ there exists a partition P of S such that $V(Q,f)+V(Q,g)-V(Q,f+g)<\varepsilon$ for any refinement Q of P. The author proves that the set of all functions syntonic with a given function forms a sublattice of the lattice L of all functions on S, which is simultaneously a convex cone in the linear space L. Further, two functions f, g of bounded variation on S are syntonic if and only if $V(f+g)=V(f)+V(g)$. Any function f of bounded variation on S can be expressed as a sum $g+h$, where g is a non-decreasing, h a non-increasing bounded function on S, and $V(f)=V(g)+V(h)$.

V. Vilhelm (Prague)

3698:

Jordan, Pascual. Quantenlogik und das kommutative Gesetz. The axiomatic method. With special reference to geometry and physics. Proceedings of an International Symposium held at the Univ. of Calif., Berkeley, Dec. 26, 1957–Jan. 4, 1958 (edited by L. Henkin, P. Suppes and A. Tarski), pp. 365–375. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. xi+488 pp. $12.00.

After an introduction relating modular lattices to the logic of quantum mechanics, the author reviews previous work [Abh. Math. Sem. Univ. Hamburg **21** (1957), 127–138; MR **19**, 524] on skew lattices. He then discusses considerations relevant to defining the notion of a distributive skew lattice.

S. Sherman (Detroit, Mich.)

3699:

Menzel, Wolfram. Über den Untergruppenverband einer Abelschen Operatorgruppe. I. m-Verbände. Math. Z. **74** (1960), 39–51.

An m-lattice (m for Menge=set) is a lattice which is isomorphic to a complete sublattice of the lattice of all subsets of a set. It is known that finite distributive lattices can be characterized as m-lattices. Furthermore [Raney, Proc. Amer. Math. Soc. **3** (1952), 677–680; MR **14**, 612], a complete lattice L is an m-lattice if and only if every element of L is a union of perfect elements. An element p in L is perfect if for every $X \subseteq L$, from $p \leqq \bigcup\{x \mid x \in X\}$ it follows that $p \leqq x$, some $x \in X$. The author gives a different characterization of m-lattices in terms of the following concepts: (i) A complete lattice L is a (D∗1)-lattice if the distributive law $a \cap \bigcup B = \bigcup(a \cap B)$ holds; (ii) an element i of L is completely join irreducible (c.j.i.) if for every $X \subseteq L$ such that $i=\bigcup X$, it follows that $i \in X$. Theorem: A complete lattice L is an m-lattice if and only if it is a (D∗1)-lattice and every element of L is a union of c.j.i.-elements. The proof is based on the fact that in a (D∗1)-lattice an element is perfect if and only if it is c.j.i.

It is also shown that an m-lattice is characterized by the partial ordering induced on the set of c.j.i. elements. Conversely, for any partially ordered set H there exists a unique m-lattice whose p.o. set of c.j.i. elements is isomorphic to H. This is used to study chain conditions in m-lattices.

J. Hartmanis (Schenectady, N.Y.)

3700:

Menzel, Wolfram. Über den Untergruppenverband einer Abelschen Operatorgruppe. II. Distributive und m-Verbände von Untergruppen einer Abelschen Operatorgruppe. Math. Z. **74** (1960), 52–65.

Let g be an Abelian group with a right operator ring Ω and let A be the lattice of all admissible subgroups of g. Using results obtained in part I of this paper [see preceding review], the author shows that A is an m-lattice if and only if A is distributive and every element of A is a union of c.j.i. elements. It is also shown that a subgroup i in A is completely irreducible if and only if i is cyclic and has exactly one maximal subgroup. A number of conditions are derived which are equivalent to the distributivity of A, for example: (i) for arbitrary a, $b \in A$ there are no nontrivial homomorphisms between $a/a \cap b$ and $b/a \cap b$; (ii) for arbitrary a, $b \in A$, $a/a \cap b \cong b/a \cap b$ implies that $a=b$. Several other interesting results are derived which hold for more restricted g and Ω.

J. Hartmanis (Schenectady, N.Y.)

3701:

Epstein, George. The lattice theory of Post algebras. Trans. Amer. Math. Soc. **95** (1960), 300–317.

The author examines a new set of axioms for Post algebras (which correspond to n-valued logics). By introducing a collection of generalized complements, the author is able to use a rather simple set of axioms, and the role of the underlying Boolean algebra is made much

more transparent. As a result, he is able to develop a representation theorem for Post algebras directly from the representation theory for Boolean algebras. He also obtains an equivalence between the completeness of his Post algebras and the completeness of the corresponding Boolean algebras. The original description of the algebra was published by Rosenbloom [Amer. J. Math. **64** (1942), 167–188; MR **3**, 262]. *B. A. Galler* (Ann Arbor, Mich.)

THEORY OF NUMBERS

See also 3734.

3702:

Sierpiński, Wacław. ★Arytmetyka teoretyczna. [Theoretical arithmetic.] With the cooperation of Jerzy Łoś. 2nd revised ed. Biblioteka Matematyczna, Tom 7. Państwowe Wydawnictwo Naukowe, Warsaw, 1959. 273 pp. zł. 32.00.

This edition differs from the first [Państwowe Wydawnictwo Naukowe, Warsaw, 1955; MR **16**, 998] as follows. Part I is augmented with a section on set-theoretical construction of a Peano model, and two other sections on axiomatic presentations of the integers and rational numbers. The elements of the set $0, 1, 2, \cdots$ are now called "natural numbers" instead of "basic numbers".

3703:

Rotkiewicz, A. Sur les nombres pairs n pour lesquels les nombres a^nb-ab^n, respectivement $a^{n-1}-b^{n-1}$, sont divisibles par n. Rend. Circ. Mat. Palermo (2) **8** (1959), 341–342.

Using some results of G. D. Birkhoff and H. S. Vandiver [Ann. of Math. (2) **5** (1904), 173–180], and N. G. W. H. Beeger [Amer. Math. Monthly **58** (1951), 553–555; MR **13**, 320], the author proves that for positive integers a, b, there exist infinitely many positive even integers n such that a^nb-ab^n is divisible by n. In addition, the existence of infinitely many even n dividing $a^{n-1}-b^{n-1}$ is proved in case $a-b$ is even. *A. Sklar* (Chicago, Ill.)

3704:

Žuravskiĭ, A. M. On a formula of Euler. Zap. Leningrad. Gorn. Inst. **36** (1958), no. 3, 3–4. (Russian)

3705:

Duncan, R. L. A topology for sequences of integers. II. Amer. Math. Monthly **67** (1960), 537–539.

The author continues his investigation [Amer. Math. Monthly **66** (1959), 34–39; MR **20** #6402] of a topology defined on the set of all increasing sequences of positive integers which have asymptotic density.

M. Brown (Princeton, N.J.)

3706:

McCarthy, P. J. Some remarks on arithmetical identities. Amer. Math. Monthly **67** (1960), 539–548.

Let k be a positive integer and let $(a, b)_k$ denote the largest integral kth power that divides both a and b. The author considers those functions of two variables $f(n, r)$ satisfying $f(n, r)=f((n, r^k)_k, r)$ for all nonnegative integers n and all positive integers r. He establishes a considerable number of arithmetical identities involving these functions, most of which are generalizations of identities of E. Cohen [Duke Math. J. **26** (1959), 165–182; MR **21** #2616].

W. H. Mills (Berkeley, Calif.)

3707:

Cohen, Eckford. Arithmetical functions associated with the unitary divisors of an integer. Math. Z. **74** (1960), 66–80.

The author considers the unitary convolution $f(n)=\sum_{d\delta=n,(d,\delta)=1} g(d)h(\delta)$ of two arithmetical functions $g(n)$ and $h(n)$. By elementary methods asymptotic estimates are found for $\sum_{n\leq x} f(n)$ in a number of special cases. From one of these an asymptotic estimate is found for the average order of the core of n, where the core $\gamma(n)$ of n is the maximal square-free divisor of n. From another special case information is obtained about the distribution of exponentially odd integers, that is, integers in whose canonical prime-power factorization all occurring exponents are odd. The precise results are $\sum_{n\leq x}\gamma(n)=\frac{1}{2}\alpha x^2+O(x^{3/2})$ and $Q^*(x)=\alpha x+O(x^{1/2}\log x)$, where $Q^*(x)$ denotes the number of exponentially odd integers less than or equal to x, and α is a constant.

W. H. Mills (Berkeley, Calif.)

3708:

Bredihin, B. M. Natürliche Dichten einiger Zahlhalbgruppen. Ukrain. Mat. Ž. **11** (1959), 137–145. (Russian. German summary)

Some results of H.-J. Kanold [J. Reine Angew. Math. **193** (1954), 250–252; MR **16**, 569] and P. Scherk [same J. **196** (1956), 133–136; MR **18**, 284] for integers and primes in the classical sense are extended to more general systems. Let G be a multiplicative semigroup of positive real numbers $\alpha\geqq 1$ $(1\in G)$ with an infinite basis $\{\omega_1, \omega_2, \cdots\}$ $(\omega_j>1)$ having no finite limit point. This means that any α can be expressed uniquely as $\alpha=\omega_1^{x_1}\omega_2^{x_2}\cdots$, where Greek letters denote members of G and the x_j are non-negative integers of which at most a finite number are non-zero. In the classical case the α are the positive integers and the ω the primes; and apart from terminology the extension is essentially that of A. Beurling [Acta Math. **68** (1937), 255–291]. Divisibility, H.C.F., the Möbius μ-function, etc., are defined in the obvious way in terms of the ω-products of the α's. The set G is said to have natural density C if

$$(1) \qquad \nu(x) \sim Cx \quad \text{as } x\to\infty,$$

where $\nu(x)$ is the number of α's in G which are $\leqq x$; and similarly for subsets of G. The author considers in particular the subsets g', g'' defined by:

$$(g') \qquad (\alpha, \mu) = 1, \quad Q(\alpha) = \delta,$$

$$(g'') \qquad (\alpha, \mu) = 1, \quad Q(\alpha) \geqq \delta,$$

where μ, δ are fixed, and $(Q(\alpha))^2$ is the largest square divisor of α (so that $Q(\alpha)=\gamma$, where γ is uniquely defined by $\alpha=\beta\gamma^2$, $\mu(\beta)\neq 0$). The main result (theorem 1) is that, if G has natural density C, then g' and g'' have, respectively, the natural densities

$$\frac{C_G}{\delta^2}\prod_{\omega|\mu}\frac{\omega}{\omega+1} \quad (\text{if } (\delta, \mu) = 1),$$

$$C\prod_{\omega|\mu}\frac{\omega-1}{\omega}-C_G\prod_{\omega|\mu}\frac{\omega}{\omega+1}\sum_{\beta<\delta,(\beta,\mu)=1}\frac{1}{\beta^2},$$

where C_G depends only on G. It is proved also (theorem 2) that, if the asymptotic formula (1) for G is replaced by

$$\nu(x) = Cx + O(\sqrt{x}),$$

then the corresponding asymptotic formulae for g'' and g''' implied by theorem 1 may be replaced by formulae involving an error $O(\sqrt{x}\cdot\log x)$. When $\delta = 1$ the set g'' is denoted by g'. The theorems are first proved for this basic case, and the other results are then derived. All arguments are of an elementary nature.

A. E. Ingham (Cambridge, England)

3709:

Daykin, D. E. Representation of natural numbers as sums of generalised Fibonacci numbers. J. London Math. Soc. **35** (1960), 143–160.

The author completes and generalizes results for Fibonacci numbers proved by the reviewer [Simon Stevin **29** (1952), 190–195; MR **15**, 401]. First, he determines all pairs of sequences (a_n), (k_n) of natural numbers, such that (a_n) is nondecreasing and the following property P holds: each natural number N has a unique representation $N = a_{i_1} + \cdots + a_{i_d}$, with $i_{\nu+1} \geqq i_\nu + k_\nu$ $(\nu = 1, \cdots, d-1)$; the possible sequences (k_ν) are given by $(h, k, k, k, \cdots)$, where $k = h$ or $h+1$. If $k_\nu = 2$ $(\nu = 1, 2, \cdots)$, then (a_n) is the sequence of Fibonacci numbers, even when (a_n) is not supposed to be nondecreasing. Further, the average number of summands in the representations of the natural numbers N with $a_n \leqq N < a_{n+1}$ (n large) is determined. The proofs are elementary, but require detailed investigations. *C. G. Lekkerkerker* (Amsterdam)

3710:

Schinzel, A.; Szekeres, G. Sur un problème de M. Paul Erdős. Acta Sci. Math. Szeged **20** (1959), 221–229.

A problem proposed by Paul Erdős is as follows. Let $a_1 < a_2 < \cdots < a_k \leqq n$ be integers such that the least common multiple of any two a's exceeds n. Prove that $\sum_{i=1}^k a_i^{-1} < 2$. A solution by R. S. Lehman [Amer. Math. Monthly **58** (1951), 345–346] showed that the upper bound 2 could be replaced by $7/6 + 1/(6n)$. Erdős later conjectured that for every $\varepsilon > 0$, there exists an n_0 such that $\sum_{i=1}^k a_i^{-1} < 1 + \varepsilon$ for $n > n_0$. The following three results are established in the present paper: (1) $\sum_{i=1}^k a_i^{-1} \leqq 31/30$, equality being attained only for $a_1 = 2$, $a_2 = 3$, $a_3 = 5 = n$; (2) for every $\varepsilon > 0$, there exists an n_0 such that for $n > n_0$, $\sum_{i=1}^k a_i^{-1} < c + \varepsilon$, where $c = 1.017262\cdots$ is explicitly given in terms of certain rational numbers c_j $(j = 1, \cdots, 58)$; (3) for every $\varepsilon > 0$, there exists an n_0 such that for $n > n_0$ and a certain sequence $\{a_i\}$, $\sum_{i=1}^k a_i^{-1} > 1 - \varepsilon$. The idea of the proof is as follows: the author shows that if there is a sequence $\{c_j\}$ such that for every integer q, $S_q = \sum_{j=1}^\infty c_j \sum_p p^{-1} \geqq 1$, where in $\sum_p$ the summation is for $q/(j+1) < p \leqq q/j$, then $\sum_{i=1}^k a_i^{-1} \leqq S_n$. He then exhibits a specific sequence $\{c_j\}$ of rational numbers (in which, incidentally, $c_j = 0$ for $j > 58$ and for a number of smaller values of j) with the required properties. Moreover $S_q < 31/30$, except for $q = 5, 13, 19, 20, 31, 32, 61, 62$.

R. D. James (Vancouver, B.C.)

3711:

Manin, Yu. I. On cubic congruences to a prime modulus. Amer. Math. Soc. Transl. (2) **13** (1960), 1–7.

The Russian original [Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 673–678] has already been reviewed [MR **18** #380].

3712:

Carlitz, L. Congruence properties of the Weierstrass Al-functions. Math. Ann. **140** (1960), 9–21.

In this paper the author derives congruences satisfied by the coefficients of the Weierstrass Al-functions, $\mathrm{Al}_s(w)$, $s = 0, 1, 2, 3$. For example, he puts

$$\mathrm{Al}(w) = \mathrm{Al}_0(w) = \sum_{r=0}^{\infty} D_{2r}(k^2)\frac{w^{2r}}{(2r)!} \quad (D_0 = 1,\ D_2 = 0),$$

$$\mathrm{sn}(w) = \frac{\mathrm{Al}_1(w)}{\mathrm{Al}(w)} = \sum_{n=0}^{\infty} a_{2n+1}(k^2)\frac{w^{2n+1}}{(2n+1)!},$$

where $D_{2r}(k^2)$ and $a_{2n+1}(k^2)$ are polynomials in k^2 with rational integral coefficients, and proves that D_{2n} satisfies

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} a_{p^{(r-s)t/(p-1)}}(k^2) D_{2n+st}(k^2) \equiv 0 \pmod{p^{r_1 e}},$$

where $p^e(p-1) \mid t$ and r_1 is the greatest integer $\leqq \frac{1}{2}(r+1)$. He shows also that D_{2n} satisfies $D_{2n+p-1} \equiv a_p D_{2n} + (2n-1)D_{p+1}D_{2n-2} \pmod{p}$. In the so-called singular case in which the period quotient belongs to an imaginary quadratic field of discriminant d he shows that, if the Legendre symbol $(d|p) = -1$, then D_{2n} satisfies $D_{2n} \equiv 0 \pmod{p^r}$ provided $2n \geqq (2r-1)p(p-1)$, $r \geqq 1$. The method of proof is based upon ideas developed by the author in four earlier papers [Duke Math. J. **20** (1953), 1–12; Math. Ann. **127** (1954), 162–169; Math. Z. **64** (1956), 425–434; Arch. Math. **10** (1959), 460–465; MR **14**, 621; **15**, 604; **17**, 1057; **22** #689].

A. L. Whiteman (Princeton, N.J.)

3713:

Carlitz, L. Note on Alder's polynomials. Pacific J. Math. **10** (1960), 517–519.

Alder's polynomial $G_{M,t}(x)$ [same J. **4** (1954), 161–168; MR **15**, 856] may be defined by means of

$$1 + \sum_{s=1}^{\infty} (-1)^s k^{Ms} x^{s((2M+1)s-1)/2}(1 - kx^{2s})\frac{(kx)_{s-1}}{(x)_s} = \prod_{n=1}^{\infty}(1 - kx^n)\sum_{t=0}^{\infty}\frac{k^t G_{M,t}(x)}{(x)_t},$$

where M is a fixed integer $\geqq 2$ and

$$(a)_t = (1-a)(1-ax)\cdots(1-ax^{t-1}), \quad (a)_0 = 1.$$

Recently V. N. Singh [ibid. **9** (1959), 271–275; MR **21** #7318] proved that $G_{M,t}(x) = x_t$ $(t \leqq M-1)$. The present author gives another proof of Singh's result and also obtains the explicit formula

$$G_{M,t}(x) = \sum_{Ms \leqq t;\, s \geqq 0} (-1)^s \frac{(x)_t}{(x)_s (x)_{t-Ms}} x^{s(s-1)/2+st}(1 - x^s + x^{t-Ms+s})$$

valid for all t. The derivation is straightforward.

A. L. Whiteman (Princeton, N.J.)

3714:

McCarthy, P. J. Some properties of the extended Ramanujan sums. Arch. Math. **11** (1960), 253–258.

The extended Ramanujan sum of Eckford Cohen [Duke

Math. J. **16** (1949), 85–90; MR **10**, 354] is defined by $c_k(n, r) = \sum \exp(2\pi i\, nt/r^k)$, where t runs over the positive integers less than r^k such that t and r^k have no common kth power divisors other than 1; $c_1(n, r)$ is the ordinary Ramanujan sum. In this paper the author extends to $c_k(n, r)$ several of the known properties of $c_1(n, r)$. For example, he proves that

$$c_k(n^k, r)c_k(r^k, n) = \phi_k((n, r))c_k((n, r)^k, [n, r]),$$

where $\phi_k(r) = c_k(0, r)$ coincides with Jordan's function. In the case $k=1$ this result is due to Venkataraman [J. Indian Math. Soc. (N.S.) **13** (1949), 65–72; MR **11**, 329].

A. L. Whiteman (Princeton, N.J.)

3715:

Makai, E. The first main theorem of P. Turán. Acta Math. Acad. Sci. Hungar. **10** (1959), 405–411. (Russian summary, unbound insert)

P. Turán's first main theorem [*Eine neue Methode in der Analysis und deren Anwendungen*, Akadémiai Kiadó, Budapest, 1953; MR **15**, 688] deals with

$$Q = \max_{m+1 \leq \nu \leq m+n} \frac{|b_1 z_1{}^\nu + \cdots + b_n z_n{}^\nu|}{|b_1 + \cdots + b_n| \min_{1 \leq j \leq n} |z_j|^\nu}.$$

He proved that, for all values of the $b_1, \cdots, b_n, z_1, \cdots, z_n$, we have $Q \geq \{n/2e(n+m)\}^n$. This was improved by I. Dancs [same Acta **9** (1958), 309–313; MR **21** #668] who proved that $Q \geq (2e)^{-1}\{n/2e(m+n)\}^{n-1}$. The present author obtains the best possible result, viz.,

$$Q \geq \left[\sum_{k=0}^{n-1} 2^k \binom{m+k}{k}\right]^{-1}.$$

In an appendix an improvement is given of Turán's second main theorem. In a previous paper [ibid. **9** (1958), 299–307; MR **21** #1293] the author showed that the absolute constant A occurring in that theorem satisfies $2e \leq A \leq 8e$. He now shows that $A \geq 2e/\log 2$.

N. G. de Bruijn (Eindhoven)

3716:

Furuta, Yoshiomi. Norm of units of quadratic fields. J. Math. Soc. Japan **11** (1959), 139–145.

Let m be an odd rational integer, d (≥ 3) a squarefree natural number, P the rational number field, and ε_0 a fundamental unit of $Q = P(\sqrt{d})$. As is easily seen, there exists a unit ε of Q, with $N\varepsilon = -1$, congruent to a rational integer mod m, if and only if for one of the odd prime divisors of d and for each prime divisor of m, in Q, the orders of the residue classes involving ε_0 are divisible by 4, not by 8. The author obtains the result in a formulation concerning a residue symbol (artificially defined) and studies some special cases [cf. Rédei, Acta Math. Acad. Sci. Hungar. **4** (1953), 31–87; MR **16**, 450].

K. Masuda (Sendai)

3717:

Zaikina, N. G. Distribution of non-residues of degree n modulo a prime ideal in an imaginary quadratic field. Moskov. Gos. Ped. Inst. Uč. Zap. **108** (1957), 273–282. (Russian)

From the introduction: "The article gives an estimate for the smallest (in absolute value) non-residue n for a prime modulus in an imaginary quadratic field. An asymptotic formula is calculated for the number of residues of degree n in regions of special type."

3718:

Dzewas, Jürgen. Quadratsummen in reellquadratischen Zahlkörpern. Math. Nachr. **21** (1960), 233–284.

The author considers the following two related problems. (A) The determination of all real-quadratic number fields in which the genus of the form $x'E_q x = x_1{}^2 + \cdots + x_q{}^2$ ($q \geq 2$) contains only one class. He shows that for $q \geq 9$ there are no such forms. For $q=8$ only the rational field has this property. For $q \geq 4$ there is only a finite number of totally real number fields with this property; they are of at most the 11th degree, and have a discriminant not greater than 62,122,500. For $q=2, 3$ there is, for any given degree, only a finite number of totally real fields with this property. More detailed results are given for $q=2$. (B) The determination of the number of representations of a number of one of these fields as a sum of q squares of integers. This is solved for the above-mentioned fields.

B. W. Jones (Boulder, Colorado)

3719:

Lavrik, A. F. On the problem of distribution of the values of class numbers of properly primitive quadratic forms with negative determinant. Izv. Akad. Nauk UzSSR. Ser. Fiz.-Mat. **1959**, no. 1, 81–90. (Russian. Uzbek summary)

Let $h(-\Delta)$ be the class number of the binary quadratic properly primitive forms of determinant $-\Delta$ where Δ is a positive integer. Denote by $Q\{\cdots\}$ the number of positive integers $\Delta \leq N$ which satisfy the conditions inside the curly brackets. The main result of the author is that

$$\lim_{N\to\infty} \frac{1}{N} Q\{\alpha(x)\sqrt{\Delta} < h(-\Delta) < \beta(x)\sqrt{\Delta}\} > \frac{5x^2-1}{5x^2},$$

where $x>0$ is a constant and where $\alpha(x) = 2\pi/7\zeta(3) - 2x/\pi$, $\beta(x) = 2\pi/7\zeta(3) + 2x/\pi$. Hence, for sufficiently large N, $Q\{0.04\sqrt{\Delta} < h(-\Delta) < 1.45\sqrt{\Delta}\} > 0.83N$ and $Q\{h(-\Delta) < 8\sqrt{\Delta}\} > 0.998N$. The proof is based on the asymptotic formulae

$$\sum_{\Delta \leq N} \{h(-\Delta)\}^2 = \frac{36}{29\pi^2} N^2 \sum_{n=1}^{\infty} \frac{\varphi(n)\tau(n^2)}{n^3} + O(N^{1.75+\epsilon}),$$

$$\sum_{\Delta \leq N} \{h(-\Delta)\}^3 = \frac{16}{5\pi^3} N^{5/2} \sum_{n=1,\, n \text{ odd}}^{\infty} \frac{\varphi(n)\tau_3(n^2)}{n^3} + O(N^{2.4+\epsilon}),$$

where $\varphi(n)$ and $\tau(n)$ have their usual meanings, and $\tau_i(n)$ denotes the number of times n can be written as a product of i integral factors ≥ 1.

K. Mahler (Manchester)

3720:

Vinogradov, A. I. The application of $\zeta(s)$ to the sieve of Eratosthenes. Amer. Math. Soc. Transl. (2) **13** (1960), 29–60.

The Russian original [Mat. Sb. (N.S.) **41** (83) (1957), 49–80, 415–416] has already been reviewed [MR **20** #3836].

3721:

Pan, Cheng-tung. Some new results in the additive prime number theory. Acta Math. Sinica **9** (1959), 315–329. (Chinese. English summary)

Let N be a sufficiently large odd integer. By Vinogradov, there is a representation $N = p_1 + p_2 + p_3$, where p_1, p_2, p_3 are odd primes. The author proves that in this

representation either of the following three conditions may be imposed: (1) $p_i = \frac{1}{3}N + O(N^{(5+12c)/(6+12c)+\varepsilon})$ $(i = 1, 2, 3)$, where $c = 15/19$, $\varepsilon > 0$; (2) $p_1 \leqq N$, $p_2 \leqq N$, $p_3 \leqq N^{2c/(1+2c)+\varepsilon}$; (3) $p_1 \leqq N^{2/3+\varepsilon}$, $p_2 \leqq N^{2/3+\varepsilon}$, $N - N^{2/3+\varepsilon} < p_3 \leqq N$. The result (1) improves one by Haselgrove [J. London Math. Soc. **26** (1951), 273–277; MR **13**, 438], and the result (2) implies that there are even numbers between N and $N + N^{2c/(1+2c)+\varepsilon}$ which are the sum of two primes.

K. Mahler (Manchester)

3722:

Narkiewicz, W. Remarks on a conjecture of Hanani in additive number theory. Colloq. Math. **7** (1959/60), 161–165.

Let $a_1 < a_2 < \cdots$ and $b_1 < b_2 < \cdots$ be two sequences of integers. Denote by $f(n)$ the number of solutions of $n = a_i + b_j$, $A(x) = \sum_{a_i \leqq x} 1$, $B(x) = \sum_{b_i \leqq x} 1$. Hanani conjectured that if $f(n) \geqq 1$ for $n > n_0$ and lim sup $A(x)B(x)/x \leqq 1$, then one of the sequences $\{a_k\}$, $\{b_k\}$ must be finite. First of all the author states the following more general conjecture: If $f(n) \geqq k$ for $n \geqq n_0$ and

$$(1) \qquad \limsup \frac{A(x)B(x)}{x} \leqq k,$$

then one of the sequences $\{a_k\}$, $\{b_k\}$ must be finite. The author proves the following result: If $f(n) \geqq k$ holds for all n, neglecting a sequence of density 0, and if (1) holds, then $f(n) = k$ for all n, neglecting a sequence of density 0, and

$$\lim_{x=\infty} \frac{A(2x)}{A(x)} = 1 \quad \text{or} \quad \lim_{x=\infty} \frac{B(2x)}{B(x)} = 1.$$

P. Erdös (Haifa)

3723:

Müller, Bruno. Einige Untersuchungen zur additiven Zahlentheorie auf mehrdimensionalen reellen Punktmengen. J. Reine Angew. Math. **203** (1960), 1–34.

Let m, $\underline{m}$, $\overline{m}$ denote the Lebesgue measure (interior, exterior), $x = (x_1, \cdots, x_r)$ an r-dimensional vector, o the zero vector, $[o, x]$ the set of all vectors $a_1, \cdots, a_r$ with $0 \leqq a_i \leqq x_i$. If A is a set of r-dimensional vectors with non-negative components put $m(A, x) = m(A \cap [o, x])$, $m(x) = m[o, x]$. For k sets $A_1, \cdots, A_k$ put

$$\sigma(A_1, \cdots, A_k) = \inf_{x>0} \left(\sum_1^k \underline{m}(A_i, x)/m(x)\right),$$

$$\sigma^*(A_1, \cdots, A_k) = \liminf \left(\sum_1^k \underline{m}(A_i, x)/m(x)\right).$$

We call σ the density, σ^* the asymptotic density of $(A_1, \cdots, A_k)$. If the limit exists then σ^* is called the natural density of $(A_1, \cdots, A_k)$. For any set B we put $g(x) = \min\,(n | x \in nB)$ if $x \in nB$ for some positive integer n; otherwise we put $g(x) = \infty$. We also put $h(x) = \liminf_{y \to x} g(y)$. We now define the basis orders of B as follows: $H = \sup(g(x))$ (exact order), $h = \sup(h(x))$ (weak order), $\Lambda = \sup_{x>0} (\int_{[o,x]} g(a)d(a)/m(x))$ (exact mean order), $\lambda = \sup_{x>0} (\int_{[o,x]} h(a)d(a)/m(x))$ (weak mean order). The asymptotic orders H^*, h^*, Λ^*, λ^* are defined by replacing in the definitions above $\sup_{x>0}$ by $\inf_{b>0} \sup_{x>b}$ and $\int_{[o,x]}$ by $\int_{[b,x]}$. The quantities Λ and Λ^* are defined only if the integrals exist, while λ and λ^* always exist.

The main results are as follows: Let $\tilde{B}$ denote the closure of B, $\sigma(B) = \beta$, $\sigma^*(B) = \beta^*$, $\sigma(\tilde{B}) = \tilde{\beta}$, $\sigma^*(\tilde{B}) = \tilde{\beta}^*$. If $\beta, \beta^*, \tilde{\beta}, \tilde{\beta}^* > \frac{1}{2}$, then $H, h, H^*, h^* = 2$, $\Lambda = 2 - \beta$, $\lambda = 2 - \tilde{\beta}$, $\Lambda^* = 2 - \beta^*$, $\lambda^* = 2 - \tilde{\beta}^*$. If $0 < \beta \leqq \frac{1}{2}$ and if B is closed, then $H \leqq r\langle 1/\beta \rangle$, $\Lambda \leqq (r - (r-1)\beta^2)/\beta$, where $\langle a \rangle$ denotes the smallest integer not exceeded by a, while if B is arbitrary then $\Lambda \leqq H \leqq 5r \ln(1/\beta)/(\beta F(r))$, where $F(r)$ is a complicated but well-defined function of r. If $0 < \tilde{\beta} \leqq \frac{1}{2}$ then $h \leqq r\langle 1/\tilde{\beta} \rangle$, $\lambda \leqq (r - (r-1)\tilde{\beta}^2)/\tilde{\beta}$. If $\sigma(A) = \alpha$ and B is a basis of weak mean order λ, and if $C = A + B$, $\sigma(C) = \gamma$, then $\gamma \geqq \alpha(1 + F(r)(1-\alpha)/\lambda)$. A similar inequality holds with α, γ, λ replaced by α^*, γ^*, λ^* if $o = (0, \cdots, 0)$ is in $\tilde{B}$. If A has natural density then $\gamma^* \geqq \alpha^*(1 + (1-\alpha^*)/2^r\lambda^*)$. The author also gives an estimate for the function $F(r)$.

H. B. Mann (Columbus, Ohio)

3724:

Hartman, S. A feature of Dirichlet's approximation theorem. Acta Arith. **5**, 261–263 (1959).

Let p be a prime. The author considers the inequalities (1) $0 < x \leqq ct$, $|\alpha x - y| < t^{-1}$, where α is irrational and x, y are integers with $p \nmid x$. He constructs an α such that for any $c > 0$ and some $t = t(c) > 1$ there is no solution of (1). Using an ergodic theorem for continued fractions proved by Ryll-Nardzewski [Studia Math. **12** (1951), 74–79; MR **13**, 757] he shows that, in the case $p = 2$, this is true for almost all α.

C. G. Lekkerkerker (Amsterdam)

3725:

Sós, Vera T. On a problem of S. Hartman about normal forms. Colloq. Math. **7** (1959/60), 155–160.

If ξ is a real number denote as usual by $\|\xi\|$ the difference between ξ and the nearest integer (taken positively). A pair of real numbers (α, β) is called normal [respectively, positively normal, negatively normal] if there exist positive constants t_0, c depending only on (α, β), such that for any $t > t_0$ there exists an integer x for which $\|x\alpha - \beta\| \leqq t^{-1}$ and $|x| < ct$ [respectively, $0 < x < ct$, $0 < -x < ct$]. The author shows that there exist pairs (α, β) which are normal but neither positively nor negatively normal, thereby answering a question of S. Hartman [same Colloq. **6** (1958), 334]. The proof uses the algorithm developed by the author [Acta Math. Acad. Sci. Hungar. **9** (1958), 229–241; MR **20** #1670], which is substantially the same as that employed by the reviewer [Math. Ann. **127** (1954), 288–304; MR **15**, 687] and by R. Descombes [Ann. Sci. École Norm. Sup. (3) **73** (1956), 283–355; MR **19**, 253].

J. W. S. Cassels (Cambridge, England)

3726:

Cugiani, Marco. Sulla approssimabilità dei numeri algebrici mediante numeri razionali. Ann. Mat. Pura Appl. (4) **48** (1959), 135–145.

The author uses methods of Roth [Mathematika **2** (1955), 1–20; corrigendum, 168; MR **17**, 242] and Schneider [*Einführung in die transzendenten Zahlen*, Springer, Berlin, 1957; MR **19**, 252; especially the theorem on page 13] to prove the following theorem. Let α be a fixed algebraic number of degree $g > 1$ and let b be a fixed natural number; suppose that there is a sequence of pairs (p_n, q_n) of coprime integers, where $q_n = q_n' q_n''$ and q_n'' is a power of b, such that the following two conditions

are satisfied: (a) there exist constants η, ω, with $0\leqq\eta\leqq 1$, $0\leqq\omega$, such that

$$\frac{\log q_{n'}}{\log q_n}\geqq 1-\eta-\frac{\omega}{(\log\log\log q_n)^{1/2}};$$

(b) there exists an ε such that

$$\left|\alpha-\frac{p_n}{q_n}\right|<q_n^{-1-\eta-f(q_n)},$$

where

$$f(q_n)=[(4+2\eta)(2g+1)+\omega+\varepsilon](\log\log\log q_n)^{-1/2}.$$

Then

$$\limsup_{n\to\infty}\frac{\log q_{n+1}}{\log q_n}=+\infty.$$

J. W. S. Cassels (Cambridge, England)

3727:

Flor, Peter. Inequalities among some real modular functions. Duke Math. J. **26** (1959), 679–682.

Let $\alpha=(a_0, a_1, a_2, \cdots)$ be the regular continued fraction of an irrational α and put $\alpha_n=(a_n, a_{n+1}, \cdots)$, $\beta_n=(a_{n-1}, \cdots, a_1)$, $M(\alpha)=\limsup(\alpha_n+\beta_n^{-1})$, $T(\alpha)=\limsup\alpha_n\beta_n$, $k(\alpha)=\limsup a_n$. The author derives and discusses some simple inequalities relating the functions $M(\alpha)$, $T(\alpha)$, $k(\alpha)$, such as $T(\alpha)^{1/2}+T(\alpha)^{-1/2}\leqq M(\alpha)\leqq T(\alpha)-T(\alpha)^{-1}$. *C. G. Lekkerkerker* (Amsterdam)

3728:

Oppenheim, A. A note on continued fractions. Canad. J. Math. **12** (1960), 303–308.

Bankier and Leighton [Amer. J. Math. **64** (1942), 653–668; MR **4**, 81] investigated periodic proper continued fractions

$$(1)\qquad y\sim b_0+\frac{a_1|}{|b_1}+\frac{a_2|}{|b_2}+\cdots,$$

where a_i, b_i are integers with $1\leqq a_i\leqq b_i$ $(i=1, 2, \cdots)$. The author gives, for arbitrary quadratic irrational y, periodic expansions of type (1), with a period of length 1. A conjecture is formulated. *C. G. Lekkerkerker* (Amsterdam)

3729:

Wild, R. E. On the number of lattice points in $x^t+y^t=n^{t/2}$. Pacific J. Math. **8** (1958), 929–940.

Let t, n be real numbers with $n>1$, $t>1$ and t of the form $2M/(2N+1)$ where M, N are positive integers. Denote by $L(n, t)$ the number of integral points (x, y) which satisfy the inequality $|x|^t+|y|^t\leqq n^{t/2}$. The author derives an expression for the integral $\int_0^n L(w, t)w^{t/2-1}dw$, accurate to within $O(n^{(t-1)/2})$, and states that his method fails to establish a relation when $0<t<1$. In case $t=2$ the result is known [Landau, *Vorlesungen über Zahlentheorie, Vol. 2*, Chelsea, New York, 1947; pp. 221–235]. For other expressions involving $L(n, t)$ see Bachmann [*Zahlentheorie, Vol. 2*, Teubner, Leipzig, 1894; pp. 447–450], Cauer [Dissertation, Gottingen, 1914], and van der Corput [Dissertation, Leyden, Noordhoff, Gröningen, 1919].

A. C. Woods (New Orleans, La.)

3730:

Mordell, L. J. Lattice octahedra. Canad. J. Math. **12** (1960), 297–302.

Let A_i $(i=1, 2, \cdots, n)$ be a basis of the real vector space R^n, and K the convex hull of $\{\pm A_i\}$. The determination of all the lattices Λ of R^n (i.e., the discrete subgroups of R^n) such that $K\cap\Lambda=\{0, \pm A_i\}$ was known for $n\leqq 4$ [E. Bruungraber, Dissertation, Wien, 1944; K. H. Wolff, Monatsh. Math. **58** (1954), 38–56; MR **16**, 341]. The author gives a simpler proof for the result.

I. G. Amemiya (Tokyo)

3731:

Linnik, Yu. V. Asymptotic-geometric and ergodic properties of sets of lattice points on a sphere. Amer. Math. Soc. Transl. (2) **13** (1960), 9–27.

The original Russian version [Mat. Sb. (N.S.) **43 (85)** (1957), 257–276] has already been reviewed [MR **20** #2328].

FIELDS

See 3717, 3718.

ABSTRACT ALGEBRAIC GEOMETRY

See also 3645, 3748.

3732:

Gauthier, Luc. L'invariant modulaire dans la géométrie sur un corps de caractéristique 3. J. Math. Pures Appl. (9) **38** (1959), 117–163.

The author deals with the theory of algebraic curves defined over a field of characteristic 3. In part I he gives a canonical form of the equation $y^2=f(x)$, where $f(x)=a_0x^4+a_1x^3+a_2x^2+a_3x+a_4$, of an elliptic curve, and gives a classification of elliptic curves by the invariant $j=(r+1)^6/r^2(r-1)^2$, where r is the cross-ratio of the roots of the equation $f(x)=0$. His result coincides with that of Deuring [Math. Z. **47** (1940), 47–56; MR **3**, 266]. In part II the author considers a non-singular algebraic curve of genus 3 given by an equation of degree 5 such that each canonical divisor has the cross ratio -1. This is the family of curves considered by F. Klein in the classical case, each of which has a group of automorphisms G_{168}. In the case of characteristic 3 such a curve has the properties: (i) it has a group of automorphisms of order 6048; (ii) the invariant of Hasse-Witt is zero; (iii) the jacobian variety is isogenous to a variety which is the direct product of three elliptic curves with $j=0$; (iv) all its points are Weierstrass points. Conversely, each of these properties characterizes the above family of curves among those of genus 3. *Y. Kawada* (Tokyo)

LINEAR ALGEBRA

See also 3955, 3956a-b.

3733:

Finkbeiner, Daniel T., II. ★Introduction to matrices and linear transformations. A Series of Undergraduate Books in Mathematics. W. H. Freeman and Co., San Francisco, Calif.-London, 1960. vii+248 pp. $6.50.

For the most part, this text is carefully and logically

written, and the proofs are complete but concise. The treatment is axiomatic, which allows the use of modern algebraic techniques. Computational considerations are not emphasized. Properties of linear transformations of vector spaces are used to motivate matrix definitions, and thereafter the two topics are developed simultaneously. The only exception to this careful presentation is in the first chapter where some definitions are informal and others are omitted. The author states that the text is suitable for "undergraduates who possess reasonable mathematical aptitude". It seems more likely to be successful with graduates or with undergraduates of exceptional ability or previous training in abstract algebra.

Chapter 1 gives definitions and brief discussion of fundamental terms and concepts. Sets, mappings, relations, abstract systems in general, groups and fields are considered. These concepts are re-defined more precisely in Appendix A. It seems to this reviewer that the inclusion of appendix A as an integral part of chapter 1 would have made the book more consistent.

Chapter 2 treats vector spaces over a field with discussion of linear dependence, basis, dimension, and isomorphism between spaces. Each of these topics is treated in complete generality. The vector space of n-tuples of elements of a field with the usual operations appears only as an illustrative example.

In chapter 3, definitions and properties of linear transformations of a vector space are introduced, followed in chapter 4 by an analogous treatment of matrices. The close relationship between the two concepts is emphasized throughout. A linear algebra over a field is defined and the theorem that the set of all $n\times n$ matrices over a field is isomorphic to the linear algebra of all linear transformations on an n-dimensional vector space over the field is proven.

In chapter 5, a determinant is defined as a function "det" whose domain is the set of all $n\times n$ matrices over a field, whose range is a subset of the field, and which (a) is a linear function of each column, (b) is zero if two columns are equal, and (c) has the value 1 on the identity matrix. The usual theorems on determinants follow. The chapter also includes a discussion of linear equations and their solutions.

Chapters 6 and 7 include traditional material on equivalence of matrices, canonical forms, characteristic vectors and values, diagonalization theorems, and some applications.

Chapter 8 on canonical forms for linear transformations contains several theorems not usually found in beginning texts on matrices. Theorems relating to decomposition of a vector space as a direct sum of subspaces, treatment of nilpotent transformations, and the Cayley-Hamilton theorem are included.

Chapter 9 on metric concepts and chapter 10 on functions of matrices contain the usual theorems, and the latter includes some application to computational methods and solution of differential equations.

J. E. Whitesitt (Bozeman, Mont.)

3734:

Daykin, David E. Distribution of bordered persymmetric matrices in a finite field. J. Reine Angew. Math. **203** (1960), 47–54.

A matrix of the type (α_{i+j-1}), $1\leq i\leq m$, $1\leq j\leq n$, is persymmetric. The author considers the problem of finding the number of persymmetric matrices with elements in a finite field which have certain specified properties. Theorem 1: The number of persymmetric $m\times n$ matrices of rank k. Theorem 2: The number of persymmetric $m\times n$ matrices M of rank k such that $\Phi(M)=u$, where if there is a least integer u such that $\alpha_u\neq 0$ then $\Phi(M)=u$, otherwise $\Phi(M)=m+n$. Theorem 3: Let A be a fixed square persymmetric matrix of order m and rank k. Determination of the number of ways in which A may be bordered on the right by s columns and below by r rows so that the resulting matrix is persymmetric and has rank $k+h$. Theorem 4: The number of square persymmetric matrices M of order m such that $\deg M=\beta$.

In each case explicit formulas are obtained.

L. Carlitz (Durham, N.C.)

3735:

Daykin, D. E. On the rank of the matrix $f(A)$ and the enumeration of certain matrices over a finite field. J. London Math. Soc. **35** (1960), 36–42.

If $\mathfrak{F}$ is a field of q elements, A a square matrix over $\mathfrak{F}$ and $\mathfrak{F}[A]$ the set of all polynomials in A with coefficients in $\mathfrak{F}$, the author finds a formula for the number of matrices of rank r in $\mathfrak{F}[A]$. *H. S. A. Potter* (Aberdeen)

3736:

Pais, Gameiro. Extension of a field of a vector space. Ciência. Lisboa No. 15/16 (1958/59), 67–70. (Portuguese)

Expository.

3737:

Krasnosel'skiĭ, M. A. On some methods of approximate calculation of the characteristic values and characteristic vectors of a positive definite matrix. Amer. Math. Soc. Transl. (2) **12** (1959), 155–162.

The Russian original [Uspehi Mat. Nauk (N.S.) **11** (1956), no. 3 (69), 151–158] has already been reviewed [MR **18**, 676].

3738:

Mirsky, L. Inequalities for certain classes of convex functions. Proc. Edinburgh Math. Soc. **11** (1958/59), 231–235.

For real n-vectors x and y, set $x\prec\prec y$ if, when their components x_i and y_i are arranged in non-ascending order to give $\bar{x}_i$ and $\bar{y}_i$, then $\sum_{i=1}^k \bar{x}_i\leq\sum_{i=1}^k \bar{y}_i$ $(1\leq k\leq n)$; $x\prec y$ if in addition $\sum_{i=1}^n \bar{x}_i=\sum_{i=1}^n \bar{y}_i$; $x\leq y$ if $x_i\leq y_i$, $1\leq i\leq n$. The author proves that $x\prec y$ if and only if $F(x)\leq F(y)$ for all convex symmetric functions F; and $x\prec\prec y$ if and only if $F(x)\leq F(y)$ for all convex symmetric functions F with the further property that $u\leq v$ implies $F(u)\leq F(v)$. The proofs use the fact that when $x\prec y$ there exists a doubly stochastic matrix D such that $x=Dy$, and that D is in the convex hull of the set of permutation matrices. Other inequalities are derived; in particular, well-known inequalities of Hardy-Littlewood-Pólya, Pólya, and Fan are deduced as special cases.

B. N. Moyls (Vancouver, B.C.)

3739:

Lewis, Daniel C., Jr.; Taussky, Olga. Some remarks concerning the real and imaginary parts of the characteristic roots of a finite matrix. J. Mathematical Phys. **1** (1960), 234–236.

For any normalizable matrix A with roots α_i, there exists a positive definite hermitian matrix G such that for any σ the roots of $\det(\sigma GA+\bar{\sigma}A*G-2\lambda G)=0$ are $\operatorname{Re}(\sigma\alpha_i)$. If A is not normalizable, for any $\varepsilon>0$ there exists such a G such that the roots differ from $\operatorname{Re}(\sigma\alpha_i)$ by not more than $|\sigma|\varepsilon$. In case A is real, G is real.

A. S. Householder (Oak Ridge, Tenn.)

3740:

Parter, S. On the eigenvalues and eigenvectors of a class of matrices. J. Soc. Indust. Appl. Math. **8** (1960), 376–388.

The matrices considered are sign-symmetric, which is to say that $a_{ij}a_{ji}\geqq 0$, with equality only when $a_{ij}=a_{ji}=0$. Then a graph can be associated with the matrix A in the obvious way, that when $a_{ij}a_{ji}\neq 0$, an arc joins points p_i and p_j. Assume the graph is a tree; it is known that all roots of A are real [J. Z. Hearon, Bull. Math. Biophys. **15** (1953), 121–141; MR **14**, 982]. The author obtains a condition that is necessary and sufficient for λ to be a multiple root of A. When $a_{ij}\neq 0$, let $B(j,i)$ be the submatrix associated with the subgraph containing p_j when the arc between p_j and p_i is deleted. The condition is that there exists an index i such that for at least three distinct values of j, λ is a root of $B(j,i)$.

A. S. Householder (Oak Ridge, Tenn.)

3741:

Mehta, M. L.; Gaudin, M. On the density of eigenvalues of a random matrix. Nuclear Phys. **18** (1960), 420–427.

The joint frequency distribution of the eigenvalues of a hermitian matrix is known for the case of elements that are independently normally distributed [D. N. Nanda, Ann. Math. Statist. **19** (1948), 45–57; MR **9**, 453; also, S. N. Roy, Sankhyā **7** (1945), 133–158; MR **7**, 317]. The authors here carry out the rather formidable integrations to obtain the marginal distribution of a single eigenvalue, which they express in terms of Hermite polynomials. In nuclear theory, the result represents the level density in heavy nuclei.

A. S. Householder (Oak Ridge, Tenn.)

3742:

Kucharzewski, M. Über die Funktionalgleichung $f(a_k{}^i)\cdot f(b_k{}^i)=f(b_\alpha{}^i a_k{}^\alpha)$. Publ. Math. Debrecen **6** (1959), 181–198.

This and the following two papers are concerned with the theorem: Let M_n be the set of n-square matrices over the real or complex field K, and suppose f is a mapping from M_n into K such that $f(AB)=f(A)f(B)$ for any pair A, B in M_n. Then $f(A)=\varphi(\det A)$, where φ is a complex-valued function such that $\varphi(uv)=\varphi(u)\varphi(v)$. The case $n=2$ was discussed by Gołąb [Ann. Polon. Math. **6** (1959/60), 1–13; MR **21** #2659]. In the present proof the author writes the matrix A as a product of elementary matrices (and, in case $\det A=0$, a singular diagonal matrix), and examines the nature of f on these matrices.

B. N. Moyls (Vancouver, B.C.)

3743:

Kuczma, Marek. Bemerkung zur vorhergehenden Arbeit von M. Kucharzewski. Publ. Math. Debrecen **6** (1959), 199–203.

The author simplifies Kucharzewski's proof [see the preceding review] by showing that one type of elementary matrices can be avoided.

B. N. Moyls (Vancouver, B.C.)

3744:

Hosszú, M. A remark on scalar valued multiplicative functions of matrices. Publ. Math. Debrecen **6** (1959), 288–289.

The author gives a very short proof of the theorem mentioned in the two preceding reviews. The proof uses the polar factorization of a matrix A and the fact that similar matrices have the same functional value under f.

B. N. Moyls (Vancouver, B.C.)

3745:

Kurepa, Svetozar. Functional equations for invariants of a matrix. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II **14** (1959), 97–113. (Serbo-Croatian summary)

Sich den Ergebnissen von M. Kucharzewski, M. Kuczma und M. Hosszú [vorangehende Referate] anschliessend und die Methode des letzteren weiterführend untersucht der Verf. Matrizenfunktionalgleichungen der Gestalten

$$(1)\qquad F(A+B)=K(G(A)G(B),\,H(A)+H(B)),$$

$$(2)\qquad F(AB)=K(G(A)G(B),\,H(A)+H(B)),$$

wo F,G,H unitär invariante ($F(U^*AU)=F(A)$ usw. mit beliebigen unitären U) für quadratische Matrizen $n$ter Ordnung mit komplexen Elementen definierte und Matrizen $m$ter Ordnung als Werte annehmende Funktionen sind, während Veränderliche und Funktionenwerte bei K alle Matrizen $m$ter Ordnung sind. Bei (1) wird bewiesen, dass $F(A)=f(\operatorname{Spur}A)$, bei (2), dass $F(A)=f(\operatorname{Det}A)$ sein muss, wo f eine Funktion einer komplexen Veränderlichen ist mit Matrizen $m$ter Ordnung als Werte, die eine Funktionalgleichung der Gestalt $f(x+y)=K(g(x)\cdot g(y),\,h(x)+h(y))$ bzw. $f(xy)=K(g(x)\cdot g(y),\,h(x)+h(y))$ erfüllt. Als Spezialfälle werden die Funktionalgleichungen

$$F(A+B)=F(A)+F(B),\quad F(A+B)=F(A)F(B),$$
$$F(AB)=F(A)+F(B),\quad F(AB)=F(A)F(B)$$

vom Verfasser eingehender betrachtet und auch bezüglich orthogonal invariante ($F(O'AO)=F(A)$ für alle orthogonale O) Funktionen von Matrizen mit lauter reellen Elementen gelöst.—Auch auf die Funktionalgleichung

$$F(A+B)+F(A-B)=2F(A)+2F(B)$$

wird bezüglich regulär invariante Funktionen ($F(S^{-1}AS)=F(A)$ für alle reguläre Matrizen S) von Matrizen mit reellen Elementen eingegangen. Diese Untersuchungen stützen sich auf ein Resultat des Verf. [Acad. Serbe Sci. Publ. Inst. Math. **13** (1959), 57–72].

{Die Lesbarkeit dieser schönen, interessanten und übrigens klar geschriebenen Arbeit wird durch manchmal spürbare Hast und durch fehlenden bzw. nur später gegebenen Erklärungen erschwert.}

J. Aczél (Debrecen)

3746:

Derwidué, L. Quelques théorèmes relatifs aux déterminants de Hurwitz et à divers autres déterminants intervenant dans la théorie des réseaux électriques et des servo-mécanismes. Actes du colloque de calcul numérique, Périgueux, 1957, pp. 19–28. Publ. Sci. Tech. Ministère de l'Air, Notes Tech. no. 80, Paris, 1959. vii + 87 pp. 1800 francs.

In general, without proof, the author presents some results about the location of zeros of polynomials defined as determinants of tridiagonal matrices with polynomials appearing in the main diagonal. Of interest in the theory of linear analogue devices is, given a polynomial $H(x)$, the number of its zeros with positive real part. To this end the author presents several new theorems which generalize former results obtained by the author himself, by E. Frank and by the reviewer. *H. Bückner* (Madison, Wis.)

3747:

Marcus Marvin; Newman, Morris. Permanents of doubly stochastic matrices. Proc. Sympos. Appl. Math., Vol. 10, pp. 169–174. American Mathematical Society, Providence, R.I., 1960.

An extended version of this paper appears in Duke Math. J. **26** (1959), 61–72 [MR **21** #3432].

ASSOCIATIVE RINGS AND ALGEBRAS

See also 3750, 3794.

3748:

Leicht, Johann. Über die zu Primidealen eines Polynombereiches gehörigen Differentialformen und -operatoren. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) **4** (1958), 1–42. (Romanian and Russian summaries)

Primarily expository, including descriptions of the derivations of $S/\mathfrak{p}S$ (where $\mathfrak{p}$ is a prime ideal in a polynomial ring $K[x_1, \cdots, x_n]$ and $S=(K[x_1, \cdots, x_n])_\mathfrak{p}$ is the corresponding local ring), the exterior algebra on n generators $dx_1, \cdots, dx_n$ over S modulo the differential ideal generated by $\mathfrak{p}$, and the duality between these two objects. After this exposition, the author notices that, in terms of the polynomial ring over S in the indeterminates d^ix_j ($j=1, \cdots, n$; $i=1, 2, \cdots$), in which there is an obvious derivation d, he can characterize the kth symbolic power $\mathfrak{p}^{(k)}$ and obtain two more or less known results: $\bigcap_k\mathfrak{p}^{(k)}=0$, and a functional determinant condition for an isolated $\mathfrak{p}$-primary component of an ideal to be $\mathfrak{p}^{(k)}$.
D. Zelinsky (Berkeley, Calif.)

3749:

Cohn, P. M. Simple rings without zero-divisors, and Lie division rings. Mathematika **6** (1959), 14–18.

Pursuing the technique he used in an earlier paper [Mathematika **5** (1958), 103–117; MR **21** #4174], the author proves that every ring without zero-divisors is embeddable in a simple ring without zero-divisors. Then, using the universal associative algebra of a Lie algebra, he proves that the Lie rings which are embeddable in Lie rings for which $[ax]=b$ is solvable for every $a\neq0$ and every b, are those with zero or prime characteristic.
W. G. Lister (Oyster Bay, N.Y.)

HOMOLOGICAL ALGEBRA

3750:

Jans, J. P. Projective injective modules. Pacific J. Math. **9** (1959), 1103–1108.

The paper considers the situation where a ring (with identity) has a sufficient supply of projective injective modules. Let R be a primitive ring with minimal right ideals. Then by p. 75 of Jacobson's *Structure of rings* [Amer. Math. Soc., Providence, R.I., 1956; MR **18**, 373] there exists a simple, faithful, projective, right [left] R-module M [N] such that M and N are dual spaces over a canonical division ring D. The author's first result is that M is R-injective if and only if $M\cong \mathrm{Hom}_D(N, D)$. As a corollary he obtains the theorem that if R is a primitive ring with minimal right ideals, then R is a simple ring with minimum condition if and only if R has both a left and a right faithful, simple, projective, injective module.

A minimal faithful left module is defined to be a faithful injective module that is a direct summand of every faithful module. Generalizing a theorem of Thrall [Trans. Amer. Math. Soc. **64** (1948), 173–183; MR **10**, 98], the author proves that if R is a right Noetherian, semi-primary ring, then R has a minimal faithful module if and only if the injective envelope of R is projective.
E. Matlis (Evanston, Ill.)

3751:

Buchsbaum, David A. Satellites and universal functors. Ann. of Math. (2) **71** (1960), 199–209.

Let T be an additive functor from the exact category $\mathfrak{A}$ to the exact category $\mathfrak{B}$. Assume that direct limits exist in the category $\mathfrak{B}$. Then the author defines the derived (or satellite) functors $\{S^nT\}$ without recourse to injectives or projectives. Let $0\to A'\to A\to A''\to 0$ be an arbitrary exact sequence in $\mathfrak{A}$. A connected sequence of functors $\{T^n\}$ from $\mathfrak{A}$ to $\mathfrak{B}$ is called universal if for any connected sequence $\{U^n\}$ and any natural transformation λ^0: $T^0\to U^0$ there is a unique extension $\lambda=\{\lambda^n: T^n\to U^n\}$ such that the diagram:

$$\begin{array}{ccc} T^n(A') & \to & T^{n+1}(A'') \\ \downarrow & & \downarrow \\ U^n(A') & \to & U^{n+1}(A'') \end{array}$$

commutes. The author shows that the connected sequence $\{S^nT\}$ is universal.

By considering universal functors the author shows that if $\mathfrak{X}$ is a topological space, and Φ is a family of paracompact supports, then $H^n(\mathfrak{X}, C)$ can be expressed as a satellite of Γ_Φ, where $H^n(\mathfrak{X}, C)$ denotes the Φ-cohomology of $\mathfrak{X}$ with coefficients in the sheaf C, and $\Gamma_\Phi(C)$ is the module of sections of C with supports in Φ. Also, if V is a projective variety, then $H^n(V, C)\cong \mathrm{Ext}^n(\mathcal{O}, C)\cong S^n\Gamma(C)$, where C is a coherent algebraic sheaf over V, $\mathcal{O}$ is the sheaf of local rings of V, and $\mathrm{Ext}^n(\mathcal{O}, C)$ and $S^n\Gamma(C)$ are defined strictly in the category of coherent algebraic sheaves. *E. Matlis* (Evanston, Ill.)

3752:

Dold, Albrecht. Zur Homotopietheorie der Kettenkomplexe. Math. Ann. **140** (1960), 278–298.

The author studies the category of chain complexes over a ring Λ. The heuristic analogy between spaces and

chain complexes is persuasively exploited according to the following catalogue. Space: chain complex; C-W complex: positive projective chain complex; homotopy group: homology group; suspension: dimension shift. The analogue of the mapping cylinder is introduced, following J. H. C. Whitehead [Bull. Amer. Math. Soc. **55** (1949), 213–245; MR **11**, 48] and the appropriate generalization of the Whitehead theorem is proved, viz., that if X and Y are positive projective complexes, $f:X\to Y$ and $Hf:HX\approx HY$, then f is a homotopy equivalence. If the restriction of projectivity is relaxed then the existence of an f with $Hf:HX\approx HY$ is no longer an equivalence relation. The equivalence relation it generates is called, in analogy with the topological notion, weak homotopy equivalence.

Still pursuing the analogy, it is seen that the role of the $K(\pi, n)$ is played by projective resolutions with a shift in dimension. In particular the cohomology of these, i.e., the groups $\mathrm{Ext}_\Lambda{}^n(A, B)$, gives rise to the (primary) cohomology operations on the category, composition of operations being the Yoneda product [Yoneda, J. Fac. Sci. Univ. Tokyo Sect I **7** (1954), 193–227; MR **16**, 947]. Furthermore, the Postnikov system of a complex is introduced, and it is shown that any complex is of the weak homotopy type of a successive extension (read "fibre bundle") of projective resolutions. These extensions are characterized by certain cohomology classes (k-invariants).

The author seems to be mistaken in supposing that the k-invariants of a nonprojective complex are uniquely defined. However, the vanishing of all the k-invariants has an unambiguous sense and leads to the notion of a complex of split (zerfallende) homotopy type. For these the author proves an elegant generalization of the Künneth theorem: if X and Y are of split homotopy type and $H\,\mathrm{Tor}_i(X, Y)=0$ for $i>0$ then

$$H_n(X\otimes Y) \approx \sum_{i+j+k=n} \mathrm{Tor}_k(H_iX, H_jY).$$

The same Postnikov invariants were also introduced by the reviewer in Ann. of Math. (2) **60** (1954), 283–303 [MR **16**, 276]. *A. Heller* (Urbana, Ill.)

GROUPS AND GENERALIZATIONS

See also 3700, B4443.

3753:

Janko, Zvonimir. Über das Rédeische schiefe Produkt vom Typ $G\odot\Gamma$. Acta Sci. Math. Szeged **21** (1960), 4–6.

A slight error in a proof by Rühs [same Acta **16** (1955), 160–164; MR **17**, 823] is corrected.

K. A. Hirsch (St. Louis, Mo.)

3754:

Kneser, M.; Świerczkowski, S. Embeddings in groups of countable permutations. Colloq. Math. **7** (1959/60), 177–179.

Every abelian group is shown to be isomorphic to a group of permutations of a set X such that every permutation displaces at most countably many elements of X. On the other hand the authors construct non-abelian groups G which are not isomorphic to such permutation groups. *N. G. de Bruijn* (Eindhoven)

3755:

Racah, G. Four dimensional orthogonal groups. Nuovo Cimento (10) **14** (1959), supplemento, 75–80.

The consequences of the theorem proved here that any element of the 4-dimensional complex orthogonal group may be written $D\Delta=\Delta D$, where D and Δ have specified form, are clearly and illuminatingly set forth [cf. Goursat, Ann. École Norm. Sup. (3) **6** (1889), 9–102; G. de B. Robinson, Proc. Cambridge Philos. Soc. **27** (1931), 37–48].

G. de B. Robinson (Toronto)

3756:

Comfort, W. W. The isolated points in the dual of a commutative semi-group. Proc. Amer. Math. Soc. **11** (1960), 227–233.

For a commutative semi-group G, let $\hat{G}$ denote the set of all semicharacters of G and let $\hat{G}$ have the topology of pointwise convergence. Applying theorems of Hewitt and Zuckerman [Trans. Amer. Math. Soc. **83** (1956), 70–97; MR **18**, 465], the author finds necessary and sufficient conditions for elements χ of $\hat{G}$ to be isolated. The conditions are algebraic ones depending only upon the sets $S(\chi)=\{x\in G:\chi(x)\neq 0\}$ and $N(\chi)=\{x\in G-S(\chi): xy=xz$ for some $y,\ z\in S(\chi)$ for which $\chi(y)\neq\chi(z)\}$. (The hypothesis core $S(\chi)\neq\Lambda$ in theorem 4.2 is not essential.)

K. A. Ross (Seattle, Wash.)

3757:

Tamura, Takayuki; Sasaki, Morio. Finite semigroups in which Lagrange's theorem holds. J. Gakugei Tokushima Univ. **10** (1959), 33–38.

Let S be a finite semigroup. It is shown that S, of order n, has no proper subsemigroup of order greater than $n/2$ if and only if the order of each subsemigroup divides n. Semigroups with this property are shown to be (except for those of order 2) completely simple with four possibilities only for the defining matrix.

G. B. Preston (Shrivenham)

3758:

Tamura, Takayuki; Sasaki, Morio; Minami, Yasuo; Noguchi, Toshio; Miki, Kenji; Shingai, Mitsuo; Nagaoka, Tsuguyo; Arai, Toshitaka; Muramoto, Katsuyo; Nakao, Mamoru; Naruo, Hiroaki; Himeda, Yoshiaki; Takami, Teruko. Semigroups of order ≤ 10 whose greatest c-homomorphic images are groups. J. Gakugei Tokushima Univ. **10** (1959), 43–64.

A semigroup S possesses a maximal commutative homomorphic (c-homomorphic) image $C(S)$ [reviewer's notation] which can be mapped homomorphically onto any other c-homomorphic image of S. If $C(S)$ consists of a single idempotent then S is said to be c-indecomposable. Let $S=S_0>S_1>S_2>\cdots>S_m>0$ be a maximal ideal series of the finite semigroup S. Let S_i/S_{i+1}, $i=0, 1, 2, \cdots, m-1$, be c-indecomposable and let S_m be completely simple. Then $C(S)=C(S_m)$ is a group. This fact is used as a basis for a construction of all semigroups with the property of the title. A full discussion and justification of the results is promised elsewhere.

G. B. Preston (Shrivenham)

3759:

Tamura, Takayuki. Note on finite semigroups which satisfy certain group-like condition. Proc. Japan Acad. **36** (1960), 62–64.

A congruence on a finite semigroup S is called homogeneous if each of its congruence classes contains the same number of elements. The author determines those finite semigroups on which every congruence is homogeneous. Proofs are promised later. The paper also contains a restatement of the results of the author and M. Sasaki on subsemigroups of a finite semigroup [#3757 above].
G. B. Preston (Shrivenham)

3760:
Sade, A. Anti-autotopie dans les quasigroupes. Ann. Soc. Sci. Bruxelles. Sér. I **74** (1960), 5–11; errata, 99.
A revised version of this article will appear later. (Letter from the author.)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 4073, 4074

3761:
Kloss, B. M. Probability distributions on bicompact topological groups. Teor. Veroyatnost. i Primenen. **4** (1959), 255–290. (Russian. English summary)
Proofs and extensions of results previously announced in Dokl. Akad. Nauk SSSR **109** (1956), 453–455 [MR **18**, 680]. The main result (theorem 12) asserts that for an arbitrary sequence $\{\mu_n\}$ of probability measures on a compact group G there exists a sequence $\{a_n\}$ of point measures such that $\lim \mu_1\mu_2\cdots\mu_n a_n$ exists. §§ 1–3 are largely introductory, limit theorems are proved in § 4, § 5 deals with some questions of rates of convergence, and § 6 treats measures on compact semigroups.
J. G. Wendel (Ann Arbor, Mich.)

3762:
Urbanik, K. Gaussian measures on locally compact Abelian topological groups. Studia Math. **19** (1960), 77–88.
For each locally compact abelian group, the author defines a class of Gaussian measures, which coincides in the case of a vector group or a torus group with the usual class of normal measures. Using the structure theory of locally compact abelian groups, he characterizes these measures, and shows that a group has one if and only if it is connected and that such a measure must be positive on non-void open sets. A central limit theorem for solenoidal groups is established.
K. deLeeuw (Princeton, N.J.)

3763:
Singh Varma, H. O. A note on covering groups. Nederl. Akad. Wetensch. Proc. Ser. A **63** = Indag. Math. **22** (1960), 297–301.
The author sketches a construction for the universal covering group $\tilde{G}$ of an arbitrary topological group G. Let U be a neighborhood of the identity in G. A U-covering of G consists of G', U', π', where $\pi': G' \to G$ induces an isomorphism (of topological local groups) $U' \to U$. By definition $\tilde{G}$ is the inverse limit of the U-coverings of G with appropriate mappings. A set V in G is called contiguous if for every neighborhood W of the identity and pair of points x, y in V, there is a finite sequence $x_0 = x, x_1, \cdots, x_n = y$ in V such that $x_i^{-1}x_{i+1} \in W$. It is stated that $\tilde{G}$ is metrizable, contiguous and locally contiguous whenever G has all of these properties. Moreover, $\tilde{G}$ is then monodrome, which means that the canonical projection $\tilde{\tilde{G}} \to \tilde{G}$ is an isomorphism.
P. A. Smith (New York)

FUNCTIONS OF REAL VARIABLES

See also 3697, 3738, B4514a, B4544.

3764:
Maak, Wilhelm. ★Differential- und Integralrechnung. 2., neubearbeitete Aufl. Vandenhoeck & Ruprecht, Göttingen, 1960. vi + 376 pp. DM 26.00.
This volume is an introduction to advanced calculus. Part I, of 146 pages, treats the basic notions for functions of one variable. Part II, of 222 pages, treats functions of several variables, including differential forms, implicit functions, line, surface and volume integrals, and the theorems of Gauss and Stokes.
L. M. Graves (Chicago, Ill.)

3765:
Inozemcev, O. I. Verallgemeinerung des sphärischen Stetigkeitsmoduls. Ukrain. Mat. Ž. **11** (1959), 155–162. (Russian. German summary)
Extending notions of A. Zygmund [Duke Math. J. **12** (1945), 47–76; MR **7**, 60] and S. Bernšteĭn [Dokl. Akad. Nauk SSSR **57** (1947), 111–114; MR **9**, 235] for functions of one variable, the author introduces a notion of spherical modulus of continuity of a function of several real variables. His definition involves the supremum of certain weighted means of the differences of the function and its integrals over a finite number of spherical surfaces. An even number $2m$ depending on the weights is determined. It is proved that if a continuous function f has the property that this spherical modulus of continuity is dominated by Mr^γ with $\gamma > 2m$, then f is polyharmonic ($\Delta^m f = 0$) and yields an entire function of zero exponential type.
R. G. Bartle (Urbana, Illinois)

3766:
Plamennov, I. Ya. On differential properties of measurable functions. Amer. Math. Soc. Transl. (2) **14** (1960), 31–54.
Translation of Mat. Sb. (N.S.) **42 (84)** (1957), 223–248 [MR **19**, 639].

3767:
Negoescu, N. Sur le lemme de F. Riesz. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **10** (1959), 165–172. (Romanian. Russian and French summaries)
Riesz proved his "rising sun" lemma, concerning the set of points x such that there exists $y > x$ with $f(y) > f(x)$, for continuous functions, and stated a more general result for bounded functions [see, e.g., Riesz and Sz.-Nagy, *Leçons d'analyse fonctionnelle*, 3d ed., 1955; MR **16**, 837; p. 6]. Here the author proves the lemma in detail for bounded functions. {Cf. also H. Kestelman, *Modern theories of integration*, Clarendon, Oxford, 1937, p. 199.}
R. P. Boas, Jr. (Evanston, Ill.)

3768:

Marcus, S. Conditions d'équivalence à une constante pour les fonctions intégrables Riemann et pour les fonctions jouissant de la propriété de Baire. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. No. 22 (1959), 59–62. (Romanian. Russian and French summaries)

The author gives elementary proofs of the facts that (1) if f is Riemann integrable and $\int_{-\infty}^{\infty}|f(x+h)-f(x)|dx=0$ for every h in a dense set then $\int_{-\infty}^{\infty}|f(x)-c|dx=0$ for some c; (2) if f has the Baire property (used in the form that the restriction of f to the complement of some set of first category is continuous) and if for every h in a dense set $f(x+h)=f(x)$ except perhaps for a set of x of first category, then f is constant except at most on a set of first category. *R. P. Boas, Jr.* (Evanston, Ill.)

3769:

Alda, Václav. On the surfaces without tangent planes. Amer. Math. Soc. Transl. (2) **14** (1960), 55–57.

Translation of Czechoslovak Math. J. **3** (78) (1953), 154–157 [MR **15**, 783].

3770:

Hoang, Tuĭ [Hoàng, Tuy]. The "universal primitive" of J. Markusiewicz. Izv. Akad. Nauk SSSR. Ser. Mat. **24** (1960), 617–628. (Russian)

Detailed exposition of results previously announced in Dokl. Akad. Nauk SSSR **126** (1959), 37–40 [MR **21** #5000]. *L. C. Young* (Madison, Wis.)

3771:

Levin, V. I.; Stečkin, S. B. Inequalities. Amer. Math. Soc. Transl. (2) **14** (1960), 1–29.

According to the translator (R. P. Boas, Jr.), "This is a translation of selected material from the appendices to the Russian edition of Hardy, Littlewood and Pólya's *Inequalities*. Some of the appendices consist of material which is already available elsewhere: such material has not, in general, been included in this translation." Excerpts from the varied contents follow.

Let F have bounded variation on $[0, 1]$ and $F(0)=0$. In order that $\int_0^1 \phi(x)dF(x)\geq 0$ for all continuous convex ϕ, it is necessary and sufficient that $F(1)=0$, $\int_0^1 F(x)dx=0$, and $\int_0^x F(u)du\geq 0$ $(0<x<1)$. Consequently,

$$\int_0^1 p(x)\phi(x)dx \leq \int_0^1 p(x)dx \int_0^1 \phi(x)dx$$

for such ϕ, provided p is non-decreasing on $[0, \frac{1}{2}]$ and $p(x)=p(1-x)$.

Several analogues of Carlson's inequality $(\sum a_n)^4 < \pi^2 \sum a_n^2 \sum n^2 a_n^2$ (summation 1 to ∞; $a_n\geq 0$).

For suitably restricted periodic f, and $1\leq s\leq r\leq\infty$, an upper bound on $\|f\|_r/\|f^{(k)}\|_s$. *C. Davis* (Providence, R.I.)

3772:

Opial, Z. Sur une inégalité. Ann. Polon. Math. **8** (1960), 29–32.

Where h is a positive number, let x be a real valued function defined and of class C^1 on the real interval $[0, h]$ such that $x(0)=x(h)=0$ and $x(t)>0$ for $0<t<h$. It follows readily from classical inequalities that

$$\int_0^h |x(t)x'(t)|\,dt \leq \frac{h}{\pi}\int_0^h x'^2(t)dt.$$

The author answers the question whether the coefficient h/π in this inequality is best possible by showing that for all such function x

$$\int_0^h |x(t)x'(t)|\,dt \leq \frac{h}{4}\int_0^h x'^2(t)dt$$

and that the coefficient $h/4$ is best possible. *T. A. Botts* (Charlottesville, Va.)

3773:

Olech, C. A simple proof of a certain result of Z. Opial. Ann. Polon. Math. **8** (1960), 61–63.

An alternative simpler proof of the result of Opial's described in the immediately preceding review. The present paper's formulation drops the restriction $x(t)>0$ for $0<t<h$. *T. A. Botts* (Charlottesville, Va.)

3774:

Džafarov, A. S. Some properties of functions of several variables. Dokl. Akad. Nauk Azerbaĭdžan. SSR **14** (1958), 499–503. (Russian. Azerbaijani summary)

S. M. Nikol'skiĭ [Trudy Mat. Inst. Steklov **38** (1951), 244–278; MR **14**, 32] introduced and studied the classes of functions of n variables such that the modulus of smoothness in L_p $(1\leq p\leq\infty)$ (or that of a higher generalized derivative, if it exists) for the separation h (in x_i) does not exceed $\varphi_i(|h|)=M_i|h|^{\alpha_i}$, $0<\alpha_i\leq 1$, $i=1, 2, \cdots, n$. The paper under review considers classes of functions defined in R_n with

$$\varphi_i(|h|) = M_i|h|^{\alpha_i}\prod_{j=1}^{q_i}(-\ln_j|h|)^{\beta_{j,i}},$$

where we use the abbreviations $\ln_1 x=\ln x$, $\ln_j x=\ln(\ln_{j-1}x)$ $(j=2, 3, \cdots)$. Imbedding theorems are given for these classes of functions in generalization of corresponding results of S. M. Nikol'skiĭ. *O. V. Besov* (RŽMat. **1959** #6755)

3775:

Nikol'skiĭ, S. M. Some properties of differentiable functions given on an n-dimensional open set. Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 213–242. (Russian)

The author extends the theory of his classes $W^{(r_1,\cdots,r_n)}_{p_1,\cdots,p_n}$ and $H^{(r_1,\cdots,r_n)}_{p_1,\cdots,p_n}$, as originally defined for the entire n-dimensional space or certain special subspaces (such as half-spaces), to arbitrary open subsets of n-dimensional space. In particular, the author's embedding theorem is now established for general open sets. *M. G. Arsove* (Seattle, Wash.)

3776:

Il'in, V. P. On theorems of "imbedding". Trudy Mat. Inst. Steklov. **53** (1959), 359–386. (Russian)

The author establishes a number of inequalities of the form $\int_D |f|^p\leq a_1$ implies $\int_R |\partial^k f/\partial x_1\cdots\partial x_r|^p\leq a_2$, where $R\subset D$, and is a region of a particular nature. Results of this nature are important in the modern theory of partial differential equations. *R. E. Bellman* (Santa Monica, Calif.)

3777:

Chen, Yung-ming. Theorems of asymptotic approximation. Math. Ann. **140** (1960), 360–407.

This paper contains many theorems with weighted norms, such as

$$\int_0^{2\pi} \varphi\{\alpha(x)\Theta(x)\}dx \leqq K \int_0^{2\pi} \varphi\{\alpha(x)|f(x)|\}dx,$$

where $\varphi(x)$ and $\alpha(x)$ are subjected to suitable conditions and $\Theta(x)=\sup_\xi \{(x-\xi)^{-1} \int_\xi^x f(t)dt\}$. These theorems are proved by using K. I. Babenko's results [Dokl. Akad. Nauk SSSR **62** (1948), 157–160; MR **10**, 249] and the interpolation theorem of Marcinkiewicz. It would be interesting if the author would discuss the relation between his results and those of V. F. Gapoškin [Mat. Sb. (N.S.) **46** (88) (1958), 359–372; MR **20** #6000]. Another group of the author's theorems are derived by only application of Marcinkiewicz's theorem. Some of them should be completed with weighted norms such as the results of I. I. Hirschman [Mem. Amer. Math. Soc. No. 15 (1955); MR **17**, 257]. *G. Sunouchi* (Evanston, Ill.)

3778:

Stein, E. M. On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz. Trans. Amer. Math. Soc. **88** (1958), 430–466.

Littlewood and Paley [Proc. London Math. Soc. (2) **42** (1936), 52–89] introduced the function

$$g(\theta) = \left(\int_0^1 (1-\rho)|\phi'(\rho e^{i\theta})|^2 d\rho\right)^{1/2},$$

where ϕ is a function analytic in $|z|<1$ whose real part on $|z|=1$ is $f(\theta)$. They showed that $\|g\|_p \leqq A_p\|f\|_p$, and $\|f\|_p \leqq A_p\|g\|_p$, $1<p<\infty$. In the last inequality it is assumed, of course, that $\int_0^{2\pi} f(\theta)=0$. Lusin [Bull. Calcutta Math. Soc. **20** (1930), 139–154] considered the function $s(\theta)=(\iint_\Omega |\phi'|^2 d\omega)^{1/2}$ where Ω is a kite-shaped region in $|z|<1$ with vertex at $e^{i\theta}$. Marcinkiewicz and Zygmund [Duke Math. J. **4** (1938), 473–485] showed that $\|s\|_p \leqq A_p\|\phi\|_p$, $0<p<\infty$, and that $\|f\|_p \leqq A_p\|s\|_p$, $1<p<\infty$. Marcinkiewicz [Ann. Soc. Polon. Math. **17** (1938), 42–50] introduced the function

$$\mu(x) = \left(\int_0^{2\pi} \frac{|F(x+t)+F(x-t)-2F(x)|^2}{t^3}\,dt\right)^{1/2},$$

where F is a primitive of f. It was shown by Zygmund [Trans. Amer. Math. Soc. **55** (1944), 170–204; MR **5**, 230] that the function g can be replaced by μ in the above inequalities and he gave a full discussion of the relations between these various functions. These results have been extended by the reviewer [Proc. Internat. Congr. Math., 1954, Amsterdam, Vol. 2, pp. 185–186, Noordhoff, Groningen, 1954; same Trans. **81** (1956), 167–194; **91** (1959), 129–138; MR **17**, 834; **21** #2715] to the case of functions analytic in a half-plane for the functions of Littlewood and Paley, and Lusin and to the functions of class $L^p(-\infty,\infty)$ for the function of Marcinkiewicz. The present paper extends these results to functions of several variables. Let x be a point in E_n and $f\in L_p(E_n)$. The author defines $\mu(x)=(\int_0^\infty (F_t(x))^2/t^3\,dt)^{1/2}$ where

$$F_t(x) = \int_{|y|\leq t} f(x-y)\frac{\Omega(y/|y|)}{y^{n-1}}\,dy$$

and $\Omega(x)$ is defined on $|x|=1$ to be continuous, in Lip α, $0<\alpha\leqq 1$, and such that $\int_{|x|=1}\Omega=0$. He shows then that $\|\mu\|_p \leqq A_p\|f\|_p$ for $1<p\leqq 2$. With different conditions on Ω, namely that $\int_{|x|=1}|\Omega|<\infty$ and $\Omega(x)=-\Omega(-x)$, he shows that $\|\mu\|_p\leqq A_p\|f\|_p$ for $1<p<\infty$. To define analogues of g and s, the Poisson integral of $f(x)$,

$$U(x,t) = \pi^{-(n+1)/2}\Gamma\left(\frac{n+1}{2}\right)t\int_{E_n}\frac{f(x-y)}{(|y|^2+t^2)^{(n+1)/2}}\,dy$$

is introduced. By grad U the author means what would commonly be referred to as the magnitude of the gradient, i.e., $(U_t^2+\sum U_{x_i}^2)^{1/2}$. He then defines

$$g(x) = \left(\int_0^\infty t\,\mathrm{grad}^2\,U dt\right)^{1/2},$$

$$s(x) = \left(\iint_{|x-y|<\eta t}\frac{\mathrm{grad}^2\,U}{t^{n-1}}\,dtdy\right)^{1/2}.$$

He shows that $\|s\|_p\leqq A_{\eta,p}\|f\|_p$ and $\|g\|\leqq A_p\|f\|_p$ for $1<p<\infty$. The "converse" inequalities are then established for g and s. In both these results it is assumed that $U(x,t)$ is a harmonic function in $(E_n\times(0,\infty))$ and $=o(1)$ uniformly in x as $t\to\infty$. s and g are defined as before and it is shown that there is an $f\in L_p(E_n)$ whose Poisson integral is U. The converse inequality for μ is not established. Instead the following result is given. Let $F_t^i(x)=\int_{|y|\leq t}(y_i/|y|^n)f(x-y)\,dy$, where $y=(y_1,\cdots,y_n)$, and let $\nu(x)=(\int_0^\infty \sum_1^n |F_t^i(x)|^2/t^3\,dt)^{1/2}$. Then if $f\in L_p(E_n)$, $1<p<\infty$, $\|f\|_p\leqq A_p\|\nu\|_p$. Principal tools in this investigation are the methods introduced by Calderón and Zygmund [Acta Math. **88** (1952), 85–139; Amer. J. Math. **78** (1956), 289–309; MR **14**, 367; **18**, 894] in their work on singular integrals and the results of Horváth [Nederl. Akad. Wetensch. Proc. Ser. A **56** (1953), 17–29; MR **14**, 747] on functions harmonic in several variables. There are many smaller results and comments in this paper which are also of considerable interest and the methods employed are themselves worthy of close attention. A small error occurs on p. 436 where the displayed equation which is second from the bottom of the page contains a divergent integral. This slip is easily remedied however.

D. Waterman (Lafayette, Ind.)

3779:

Kudryavcev, L. D. On implicit functions. Amer. Math. Soc. Transl. (2) **12** (1959), 137–139.

The Russian original [Uspehi Mat. Nauk (N.S.) **9** (1954), no. 3 (61), 155–156] has already been reviewed [MR **16**, 121].

3780:

Rymarenko, B. A.; Podol'nyĭ, I. P. On an extremal problem for some monotonely increasing functions. Ukrain. Mat. Ž. **11** (1959), 217–220. (Russian)

Let y_{2n} stand for a polynomial of degree $2n$, nonnegative on the real axis; let G_{2n} be the class of functions of the form

$$g_{2n}(x) = \int_{-\infty}^x e^{-z^2}y_{2n}(z)dz.$$

The author investigates functions for which $g_{2n}(\infty)$ is as small as possible, under the assumption that y_{2n} takes prescribed values s_k^2 at p prescribed points ξ_k where

either (1) ξ_k are roots of the Hermite polynomial of degree n or (2) ξ_k are arbitrary different real numbers. In case (1) the extremal y_{2n} can be given explicitly in terms of Hermite polynomials. In case (2) the extremal $g_{2n}(\infty)$ is asymptotically

$$\frac{\pi}{2m^{1/2}}\sum_{i=1}^{p}\frac{\exp(-\xi_i^2)}{\cos 4m^{1/2}\xi_i}s_i^2,$$

where $m=[n/2]$. *R. P. Boas, Jr.* (Evanston, Ill.)

3781:

McLeod, J. B. On four inequalities in symmetric functions. Proc. Edinburgh Math. Soc. **11** (1958/59), 211–219.

Let α be the set of non-negative real numbers $\alpha_1, \cdots, \alpha_n$. Denote by $c_r(\alpha)$ and $h_r(\alpha)$ the elementary and complete symmetric functions of degree r in $\alpha_1, \cdots, \alpha_n$, with the convention that $c_0(\alpha)=h_0(\alpha)=1$. Further, let r be a positive integer and s a non-negative integer. The four inequalities mentioned in the title of the paper relate to concavity and convexity (in the wide sense) of various functions of $\alpha_1, \cdots, \alpha_n$ and may be stated as follows: (I) $\{c_r(\alpha)\}^{1/r}$ is concave; (II) $\{c_{r+s}(\alpha)/c_s(\alpha)\}^{1/r}$ is concave; (III) $\{h_r(\alpha)\}^{1/r}$ is convex; (IV) $\{h_{r+s}(\alpha)/h_s(\alpha)\}^{1/r}$ is convex. It is, of course, obvious that (I) is a special case of (II), and (III) a special case of (IV). The author establishes the first three statements and conjectures the validity of the fourth. The details of the proof are fairly complicated and the argument depends, among other things, on results of Sylvester and Boole on unsignant determinants.

Statement (I) had been proved earlier, in an entirely different way, by M. Marcus and L. Lopes [Canad. J. Math. **9** (1957), 305–312; MR **18**, 877]. It may also be mentioned that J. N. Whiteley [Mathematika **5** (1958), 49–57; MR **20** #1739] found an inequality of which (I) and (III) are special cases. *L. Mirsky* (Sheffield)

MEASURE AND INTEGRATION

See also 3927.

3782:

Marcus, S. Sur la superposition de deux fonctions intégrables au sens de Riemann et sur le changement de variable dans l'intégrale de Riemann. Rev. Math. Pures Appl. **4** (1959), 381–389. (Russian)

I. D. Zaslavskiĭ [Vestnik Leningrad. Univ. **1953**, no. 11, 49–55; MR **17**, 721] defined the class R_* of real functions on an interval $[p, q]$ as consisting of those functions $\varphi(t)$ which have the property that for any function $f(x)$ defined and integrable Riemann on the interval $[a, b]$, where $a=\inf\varphi(t)$, $b=\sup\varphi(t)$ $(p\leq t\leq q)$, the composite function $f[\varphi(t)]$ will be integrable Riemann on $[p, q]$. The present paper is concerned principally with conditions on the derivative of φ which insure that φ belong to R_*. Example: If φ has a derivative everywhere, φ' is integrable Riemann, and the set of zeros of φ' are of measure zero, then $\varphi\in R_*$. Some of these results are closely connected with those of the paper of Zaslavskiĭ, unavailable to the author except through the review journals. Application is made to theorems on change of variable in Riemann integrals. *W. R. Transue* (Paris)

3783:

Hewitt, Edwin. Integration by parts for Stieltjes integrals. Amer. Math. Monthly **67** (1960), 419–423.

The author gives a general formula for integration by parts for the Lebesgue-Stieltjes integral. Let $\mathscr{B}([a, b])$ denote the collection of Borel sets in the finite closed interval $[a, b]$, and let μ and ν denote measures on $\mathscr{B}([a, b])$, where "measure" means a nonnegative, finite and countably additive set function. If $a\leq t\leq b$, let

$$M(t) = \tfrac{1}{2}\{\mu([a, t])+\mu([a, t[)\},$$

where $[a, t[$ is $[a, t]$ with the point t deleted. Define $N(t)$ similarly in terms of ν. Then the formula for integration by parts takes the form

$$\int_{[a,b]} M(t)d\nu(t)+\int_{[a,b]} N(t)d\mu(t) = \mu([a, b])\nu([a, b]).$$

The proof uses a version of Fubini's theorem that may easily be proved by Weierstrass's polynomial approximation theorem for two variables. An extension is given to measures defined for Borel subsets of the entire real line. *T. M. Apostol* (Pasadena, Calif.)

3784:

Davies, Roy O.; Marstrand, J. M.; Taylor, S. J. On the intersections of transforms of linear sets. Colloq. Math. **7** (1959/60), 237–243.

Where h is a measure function in the sense that h is an increasing real function of non-negative reals such that $h(0)=\lim_{x\to 0}h(x)=0$, a set E on the real line is said to be of h-measure zero if for every positive ε there is a decomposition $E=E^1+E^2+\cdots$ such that $\sum_{i=1}h(\operatorname{diam}E^i)<\varepsilon$. It is shown that for each measure function h there are sets $E_1, \cdots, E_4$ of h-measure zero such that: (1) the set E_1 is an F_σ set such that for every countable set $\{\varphi_1, \varphi_2, \cdots\}$ of affine transformations of the real line onto itself, the set $\bigcap_{s=1}^{\infty}\varphi_s(E_1)$ is non-empty; (2) the set E_2 is closed and such that for every finite set $\{\varphi_1, \cdots, \varphi_n\}$ of affine transformations of the real line onto itself, the set $\bigcap_{s=1}^{n}\varphi_s(E_2)$ is non-empty; (3) the set E_3 is a bounded F_σ set with the property that there exists a positive constant η such that for every countable set of reals $\{a_1, a_2, \cdots\}$ with diameter $<\eta$, the set $\bigcap_{s=1}^{\infty}\{x+a_s : x\in E_3\}$ is non-empty; (4) the set E_4 is bounded and closed and such that for every positive integer n there is a positive constant η_n such that for every set of n reals $\{a_1, \cdots, a_n\}$ of diameter $<\eta_n$, the set $\bigcap_{s=1}^{n}\{x+a_s : x\in E_4\}$ is non-empty. Result (1) implies the existence of an F_σ set E of Lebesgue measure zero such that for every countable set of reals $\{a_1, a_2, \cdots\}$, the set $\bigcap_{s=1}^{\infty}\{x+a_s : x\in E\}$ is non-empty, solving a problem posed by E. Marczewski [Colloq. Math. **3** (1954), 75]. *T. A. Botts* (Charlottesville, Va.)

3785:

Grušin, V. V. On a sufficient condition for compactness of a family of continuous functions. Uspehi Mat. Nauk **14** (1959), no. 4 (88), 165–168. (Russian)

Let $\{f\}$ be an infinite family of continuous functions defined on a closed interval $[a, b]$ and uniformly bounded by M. Further let $\nu_f(y)$ denote the number of roots of the equation $f(x)=y$; and let F be a function such that

$F(0)=0$, $F(t)>0$ for $t>0$, $F(t)\to\infty$ as $t\to\infty$, and such that for each f we have

$$(*)\qquad \int_{-M}^{M} F(v_f(y))\,dy \leq M;$$

then one can extract a sequence from the family which converges for each point in the interval. In the case that $F(t)=t$ the integral in (*) yields the total variation of f over the interval and this result reduces to the classical Helly selection theorem. *R. G. Bartle* (Urbana, Ill.)

3786:

Procenko, D. F. On a property of an invariant measure. Soobšč. Akad. Nauk Gruzin. SSR **22** (1959), 519–520. (Russian)

If E is any set, $\mathfrak{E}$ the class of all subsets of E, $f(e)$, $e\in\mathfrak{E}$, a finite set function, let $U(f,E)$, $V(f,E)$ be the Sup and Inf of the sums $\sum|f(e_k)|$ for all countable sequences $[e_k]$ of disjoint sets $e_k\in\mathfrak{E}$. The author is interested in conditions assuring that $0<V[U(f,E),E]<\infty$, or $0<U[V(f,E),E]<\infty$; in particular, for functions $\mu(e)$ defined in a Banach space, completely additive and invariant with respect to "motions".

L. Cesari (Ann Arbor, Mich.)

3787:

Shapiro, Harold N. Extensions of the Khinchine-Wisser theorem. Comm. Pure Appl. Math. **13** (1960), 15–34.

The author proves among others the following theorem: Let $\{A_n\}$ be an infinite sequence of measurable sets in a measure space E with measure μ, $\mu(E)=1$. Assume $\mu(A_n)\geq d>0$ for all n. Then to every ε there exists an infinite subsequence $\{A_{n_i}\}$ so that

$$(1)\qquad \mu(A_{n_{i_1}}\cap\cdots\cap A_{n_{i_k}}) > (1-\varepsilon)\prod_{r=1}^{k}\mu(A_{n_{i_r}})$$

holds for any k $(1\leq k<\infty)$ and any subsequence of the A_{n_i}. Several sharpenings of (1) are also proved. But the author shows that on the right side of (1) the factor $1-\varepsilon$ cannot in general be replaced by $1-\varepsilon_k$ where $\varepsilon_k\to 0$ as $k\to\infty$. [See also the paper of G. G. Lorentz reviewed below.] *P. Erdös* (Budapest)

3788:

Lorentz, G. G. Remark on a paper of Visser. J. London Math. Soc. **35** (1960), 205–208.

The author proves the following theorem which sharpens a previous result of L. Sucheston [same J. **34** (1959), 386–394; MR **21** #6608]. Let E be a measure space with measure μ, $\mu(E)=1$, and let $M(f)$ be the corresponding integral. Let $f_n(x)$, $1\leq n<\infty$, be a sequence of measurable functions satisfying $0\leq f_n(x)\leq 1$. C. Visser [Nederl. Akad. Wetensch. Proc. **40** (1937), 358–367; see also J. Gillis, same J. **11** (1936), 139–141] showed that if $M(f_n)\geq\alpha>0$ then for each $s\geq 1$ and each $\varepsilon>0$ there exists a subsequence g_n of f_n with $M(g_{n_1}\cdots g_{n_s})\geq(1-\varepsilon)\alpha^s$. The author now proves: Let $\liminf_{n_i\to\infty} M(f_{n_1}\cdots f_{n_r})=\alpha^r$, $r>0$. Then for each $\varepsilon>0$ there exists a subsequence g_n of f_n such that $M(g_{n_1}\cdots g_{n_s})\geq(1-\varepsilon)\alpha^s$, $s=r,r+1,\cdots$. [See also H. Shapiro, preceding review.] *P. Erdös* (Haifa)

3789:

Trebukova, N. I. Metric convergence and metric isomorphism. Uspehi Mat. Nauk **15** (1960), no. 2 (92), 195–199. (Russian)

Let Δ denote the measure space consisting of the interval $(0,1)$ with Lebesgue measure, and M be a measure space which is an image of Δ under an isomorphism (one-to-one mapping onto, such that it and its inverse are measure preserving). The author shows that for each sequence $f_1, f_2, \cdots$ of measurable functions on M converging in measure to a function f, there exists a sequence $U_1, U_2, \cdots$ of automorphisms of M (isomorphisms of M onto M) such that the sequence of functions $f_1(U_1x), f_2(U_2x), \cdots$ converges to f almost everywhere. *W. R. Transue* (Gambier, Ohio)

3790:

Silverman, E. A miniature theory of Lebesgue area. Amer. Math. Monthly **67** (1960), 424–430.

The Weierstrass line integral $I(f,x)$ for positive convex integrand f and rectifiable parametric curves $x=x(t)$, $0\leq t\leq 1$, $x(t)\in E_n$, has been often treated after K. Menger as a generalized length. The author shows how the main properties of $I(f,x)$ (L. Tonelli) can be elegantly obtained —for positive convex f— by the use of the same process which has been used for area in recent years. In particular the author obtains a new proof of the property of lower semicontinuity. *L. Cesari* (Ann Arbor, Mich.)

3791a:

Vinti, Calogero. La non convergenza in area delle medie integrali nel caso parametrico. Rend. Circ. Mat. Palermo (2) **8** (1959), 228–240.

3791b:

Vinti, Calogero. Espressioni che danno l'area di una superficie $z=f(x,y)$ in relazione al passaggio al limite sotto il segno. Ann. Scuola Norm. Sup. Pisa (3) **14** (1960), 103–132.

These two papers concern the possibility of expressing the area of a surface as a limit of an appropriate double integral [cf. Young, Duke Math. J. **11** (1944), 43–57; MR **6**, 121; Radó, ibid. **11** (1944), 487–496, 497–506; MR **6**, 121; Mickle, Rivista Mat. Univ. Parma **1** (1950), 197–206; MR **12**, 169]. In the first paper, given a representation of a parametric surface of finite area A on a rectangle, the author defines three expressions, of the type of lower limits of mean-value areas, and shows by an example that all three may be infinite even when $A=0$. In the second paper, the author discusses the non-parametric case. Let $z=f(x,y)$ be a continuous non-parametric surface of finite area A, defined on a rectangle $R=I\times J$ of the (x,y)-plane. Let $X=X(x,y)$ denote the partial derivative of f with respect to x, let $Y(k)=Y(k;x,y)=k^{-1}\{f(x,y+k)-f(x,y)\}$ and let $Y(h,k)=Y(h,k;x,y)$ denote the mean value in $0\leq u\leq 1$ of the expression $Y(k;x+hu,y)$. The author shows that a necessary and sufficient condition in order that A be equal to either, or both, the expressions

$$\lim_{hk}\iint_R \{1+X^2+Y^2(h,k)\}^{1/2}\,dxdy,$$

$$\lim_k \iint_R \{1+X^2+Y^2(k)\}^{1/2}\,dxdy$$

is that $f(x, y)$ be an absolutely continuous function of $x \in I$, for almost every $y \in J$.

L. C. Young (Bloomington, Ind.)

3792:

Mickle, Earl J. On a decomposition theorem of Federer. Trans. Amer. Math. Soc. **92** (1959), 322–335.

Let Λ be a Borel regular Carathéodory outer measure in euclidean space R^3 with the property that for any E with $\Lambda(E)<\infty$ there exists a null set Z of the unit sphere such that, for every $E^*\subset E$ and direction $P \not\subset Z$, $\Lambda(E^*)$ is no less than the Lebesgue measure of the planar projection of E^* parallel to P. Such measures Λ are said to satisfy the weak projection property. It is shown that there is a smallest measure μ with the weak projection property. If A is Λ-measurable and $\Lambda(A)<\infty$, then A is the union of disjoint Λ-measurable sets A_1 and A_2, where A_1 is countably 2-rectifiable and $\mu(A_2)=0$. This decomposition theorem is a modification of one by Federer [same Trans. **62** (1947), 114–192; MR **9**, 231; Thm. 9.6] for measures satisfying the (ordinary) projection property, in which A is split into three sets rather than two. If $\mu(A)<\infty$, then $\mu(A)$ equals the integralgeometric Favard 2-measure of A; it is not known whether this is so for all sets A.

W. H. Fleming (Providence, R.I.)

FUNCTIONS OF COMPLEX VARIABLES

See also 3778, 3832, 3892, 3939.

3793:

Churchill, Ruel V. ★Complex variables and applications. 2nd ed. McGraw-Hill Book Co., Inc., New York-Toronto-London, 1960. ix+297 pp. $6.75.

The first edition of this book [McGraw-Hill, New York, 1948; MR **10**, 439] was reviewed as follows: "An elementary text for a one semester course. Applications are restricted to the evaluation of integrals by residues and to certain uses of conformal mapping." The second edition is 80 pages longer than the first. Part of this additional space is devoted to a new chapter on "Integral formulas of Poisson type", and part to new theorems or proofs of theorems which had been omitted before. Even though the book, in its present form, is probably too long to be covered in a one semester course, it is nevertheless so sketchy in its development of the theory that, in the reviewer's opinion, it does not give an adequate introduction to complex variable theory. For example, the concept of the complete analytic function is not introduced at all (and hence singularities have to be treated in an extremely superficial manner), nor is the radius of convergence of a power series. Univalent functions are not mentioned nor is the Riemann mapping theorem. In view of the fact that most of the early development of the theory is based on results concerning real functions of real variables, with which the reader is assumed to be familiar, it is not clear what the author meant when he wrote in the preface to the book that the presentation is intended to be self-contained. Similarly, at least some readers will conclude that the intention of the author to give a rigorous presentation was not realized.

W. J. Thron (Boulder, Colo.)

3794:

Kuipers, L.; Scheelbeek, P. A. J. Zeros of functions of a quaternion variable. Nederl. Akad. Wetensch. Proc. Ser. A **62** = Indag. Math. **21** (1959), 496–501.

The authors prove a number of theorems concerning the zeros of certain rational functions of a quaternion variable $q=w+xi+yj+jk$. First, they give a new proof of Hamilton's theorem that the polynomial $a_0q^n+a_1q^{n-1}+\cdots+a_n$ $(n\geq 3)$ has at least one zero if the multiplication of the a_p with one another is commutative. The reasoning is similar to Littlewood's proof of the fundamental theorem of algebra. Also they prove that the form $\sum_1^p m_n/(q-q_n)$, where m_n are real numbers, has $p-1$ zeros in a plane V of the four dimensional space of (x, y, z, w) if all the points q_n lie in V. This result is somewhat analogous to Rolle's theorem.

M. Marden (Milwaukee, Wis.)

3795:

Morev, I. A. A class of monogenic functions. Mat. Sb. (N.S.) **50 (92)** (1960), 233–240. (Russian)

The functions in question are "bicomplex" functions of the form $f=\phi e_1+\Phi e_2$, where ϕ, Φ are ordinary complex functions of n variables $(n>2)$ and e_1, e_2 are the basis elements of an algebra over the complex numbers with $e_1\cdot e_1=e_1$, $e_2\cdot e_2=e_2$, $e_1\cdot e_2=e_2\cdot e_1=0$. The author introduces a concept of functions monogenic with respect to p given functions [for $p=1, 2$, cf. the author's paper, Mat. Sb. (N.S.) **42 (84)** (1957), 197–206; MR **19**, 537], and establishes differential properties and a Taylor expansion. As an application he proves the following theorem on differential equations. Let D be a domain in $E^n(x^1, \cdots, x^n)$,

$$\frac{\partial H}{\partial x_k} = \sum_{j=1}^{p} a^j \frac{\partial h^j}{\partial x_k} \quad (k=1, \cdots, n;\ 2p<n),$$

where H, a^j, h^j are complex continuously differentiable functions, and with $h^j=u^j+iv^j$,

$$\frac{\partial(u^1, \cdots, u^p, v^1, \cdots, v^p)}{\partial(x^1, \cdots, x^{2p})} \neq 0.$$

Then in a neighborhood of each $M\in D$ we have $H=F(h^1, \cdots, h^p)$, where F is holomorphic and $\partial F/\partial h^j=a^j$.

E. S. Pondiczery (Nancago)

3796:

Fedorov, V. S. A form of hypercomplex monogenic functions. Mat. Sb. (N.S.) **50 (92)** (1960), 101–108. (Russian)

The hypercomplex functions in question are sets of analytic functions from a domain D in the space of n complex variables to an associative commutative algebra A (with unit E) over the complex numbers. If f and ϕ are two hypercomplex functions, put $\delta=f_{x_1}\phi_{x_2}-f_{x_2}\phi_{x_1}$ and suppose that δ has an inverse at every point of D. If F is a given hypercomplex function, determine F_1 and F_2 from $F_{x_p}=F_1f_{x_p}+F_2\phi_{x_p}$, $p=1, 2$, and suppose that this equation continues to hold for $p=1, \cdots, n$; then F is called monogenic with respect to f and ϕ and is said to belong to class (f, ϕ, D, A). Similarly if $F_{x_p}=F_1f_{x_p}$ for $p=1, \cdots, n$ then $F\in(f, D, A)$. The functions F_1, F_2 are called the partial derivatives of F with respect to f, ϕ; they also belong to (f, ϕ, D, A). Now let $\nabla^2=\sum_{k=2}^{4}\partial^2/\partial x_k^2$, $L=\nabla^2-\lambda^{-2}\partial^2/\partial x_1^2$. The author uses hypercomplex functions to solve $LU=0$ when $n=4$, $t=x_1$,

$x=x_2$, $y=y_3$, $z=x_4$, $D=D_0\times T$, where D_0 is a region in (x, y, z)-space and T is an interval in t-space. He constructs an algebra $\bar{A}$ consisting of ordered pairs of elements of A in the same way that the complex numbers are constructed from real numbers, with $(0, E)=j$. Let $\tau=u+vj$, $\sigma=iw-\lambda j$, where $\nabla u\cdot\nabla v=0$, $(\nabla u)^2=(\nabla v)^2$, $\nabla^2 u=\nabla^2 v=0$, $(\nabla w)^2=1$, $\nabla w\cdot\nabla v=0$, $\nabla w\cdot\nabla u=0$. (1) If $f\in(\sigma, D, \bar{A})$, $\phi\in(\tau, D, \bar{A})$, $F\in(f, \phi, D, \bar{A})$, then if w is a linear function of x, y, z we have $LF=0$ and, for $w=r$, $L(F/r)=0$. (2) Let $\alpha\in(w+\lambda, D, A)$, $\beta\in(w-\lambda, D, A)$, $p\in(u-iv, D, A)$, $q\in(u+iv, D, A)$, $P\in(\alpha, p, D, A)$, $Q\in(\beta, q, D, A)$. Then if w is a linear function of (x, y, z) and P and Q satisfy $LU=0$, then P/r and Q/r satisfy the same equation for $w=r$. *E. S. Pondiczery* (Nancago)

3797:

Stečkin, S. B. An extremal problem for polynomials. Amer. Math. Soc. Transl. (2) **14** (1960), 173–180.

Translation of Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 765–774 [MR **18**, 728].

3798:

Talbot, A. The number of zeros of a polynomial in a half-plane. Proc. Cambridge Philos. Soc. **56** (1960), 132–147.

It is well known that the number of zeros of a complex polynomial in a half-plane may be determined by a continued fraction expansion, or H.C.F. algorithm. Determinantal formulae for the relevant elements of the algorithm can be obtained, and these lead to determinantal criteria for the number of zeros.

The object of the paper is to derive formulae based on the algorithm by simplex methods. In particular, methods are given for determining the degrees and leading coefficients of the algorithm quotients through the values of a series of determinants, and for determining the number of zeros of a real polynomial in the right and left half-planes, in cases when Routh's algorithm would break down. *A. Edrei* (Syracuse, N.Y.)

3799:

Watson, G. N. A theorem on continued fractions. Proc. Edinburgh Math. Soc. **11** (1958/59), 167–174.

The author provides a new proof of the Bauer-Muir theorem for continued fractions [see, for example, Perron, *Die Lehre von den Kettenbrüchen, Bd. 2*, 3te Aufl., Teubner, Stüttgart, 1957; MR **19**, 25; p. 25]. This proof has the advantage of greater symmetry. *W. J. Thron* (Boulder, Colo.)

3800:

Ullman, Joseph L. Studies in Faber polynomials. I. Trans. Amer. Math. Soc. **94** (1960), 515–528.

Let $(F_n(\zeta))$ be Faber polynomials associated with $g(w)=w+\sum_0^\infty b_n w^{-n}$ $(|w|>\rho)$, so that $g'(w)/(g(w)-\zeta)=\sum_0^\infty w^{-n-1}F_n(\zeta)$. Let $\Delta_n=\{z_{ni}: i=1, 2, \cdots, n\}$ be the set of zeros of F_n and let $\Delta=\lim\sup\Delta_n$ ($z\in\Delta$ if and only if every neighbourhood of z meets Δ_n for an infinity of n; the author's description of Δ as the derived set of the z_{ni} seems to conflict with the usual terminology). Let ρ_ζ be the maximum modulus of the zeros of $g(w)-\zeta$ in $|w|>\rho$ and let n_ζ be the number of such zeros of modulus ρ_ζ (counted according to multiplicity). Let C_n be the set of ζ for which $n_\zeta=n$. Theorem: Δ contains the frontier of C_1 but does not meet C_1.

For the case where $g(w)$ is a branch of $w(1+w^{-2})^{1/2}$, a more precise result is obtained: the frontier of C_1 is the lemniscate Γ defined by $|z^2-1|=1$, and the zeros of F_{2n+1} are distributed round Γ conformally (in a sense which need not be explained here). *W. F. Newns* (Liverpool)

3801:

Macintyre, A. J. Size of gaps and region of overconvergence. Collect. Math. **11** (1959), 165–174.

Let D be a simply connected schlicht domain which contains the unit disk as a proper subset, and let F be a closed subset of D. The author observes that if the power series $\sum c_n z^n$ has sufficiently long blocks ($n_k\leq n\leq N_k$; $N_k/n_k>\lambda(D, F)>1$) of zero coefficients, and if it represents a function which is holomorphic in D, then it should exhibit the phenomenon of overconvergence throughout F. He establishes his conjecture for the special case where D is either the whole plane cut radially from the point $z=1$ to ∞, or one of a family of domains bounded by certain portions of logarithmic spirals. His method is based on an interpolation $c_n=G(n)$, where G is an appropriate entire function. The method permits a weakening of the hypothesis in various directions; for example, the conclusion holds if λ is large enough and if the "occasional density" of zero coefficients in the blocks $n_k\leq n\leq N_k$ is sufficiently near unity. *G. Piranian* (Ann Arbor, Mich.)

3802:

Przeworska-Rolewicz, D. Sur l'intégrale de Cauchy pour un arc fermé à l'infini. Ann. Polon. Math. **8** (1960), 155–171.

The author studies the Cauchy integral $U(z)=\int_L u(\tau)/(\tau-z)\,d\tau$ when L is a curve extending to infinity. If the curve L is such that, for any z_0 not on L, the mapping $w=1/(z-z_0)$ carries L into a regular closed curve, then it is shown that, for suitable functions $u(\tau)$, $U(z)$ possesses a limit as $z\to\infty$. Application is made to the non-linear integral equation $u(t)=\lambda\int_L k[t, \tau, u(\tau)]/(\tau-t)\,d\tau$, where L is an infinite curve, and also to Hilbert problems for infinite curves. *R. C. MacCamy* (Pittsburgh, Pa.)

3803:

Carleson, Lennart. A representation formula for the Dirichlet integral. Math. Z. **73** (1960), 190–196.

Let $f(z)$ be analytic in $|z|<1$ with a finite Dirichlet integral: $D(f)=\iint_{|z|<1}|f'(z)|^2\,dxdy$. The author obtains a simple explicit evaluation of $D(f)$ involving the zeros of $f(z)$ and its boundary values. *A. Edrei* (Syracuse, N.Y.)

3804:

Gahov, F. D.; Mel'nik, I. M. Singular contour points in the inverse boundary problem of the theory of analytical functions. Ukrain. Mat. Ž. **11** (1959), 25–37. (Russian. English summary)

The 'interior inverse boundary value problem of function theory' is the following.

Given is a simple closed curve L: $w=w(s)=u(s)+iv(s)$ $(0\leq s\leq l)$ in the w-plane. It is required to find a curve C,

parametrised by its arc length s, bounding a, possibly many-sheeted, bounded domain D such that $w(s)$ is the boundary value at the point s of C of a function mapping D conformally on the interior of L.

The authors discuss the geometrical nature of D near a boundary point at which $w(s)$ has singularities of a simple type (e.g., $w(s)=|s-s_0|^r \cdot w_1(s)$, $w_1(s)$ sufficiently regular).

W. H. J. Fuchs (Ithaca, N.Y.)

3805:

Simonenko, I. B. Riemann's boundary value problem with a continuous coefficient. Dokl. Akad. Nauk SSSR **124** (1959), 278–281. (Russian)

Let C be a contour consisting of $m+1$ simple closed curves $C_0, C_1, \cdots, C_m$ bounding the connected region D^+, and denote the planar complement of $D^+ + C$ by D^-. Let $\Phi^+(z)$, $\Phi^-(z)$ be analytic in D^+ and D^-, respectively, and denote by $\Phi^+(t)$, $\Phi^-(t)$ their limiting values as $z \to t \in C$ for z in D^+, D^-, respectively. The Riemann boundary problem consists in determining the functions $\Phi^\pm(z)$ so that on C

$$(1) \qquad \Phi^+(t) = G(t)\Phi^-(t)+g(t),$$

where $G(t)$, $g(t)$ are given functions satisfying a Hölder condition on C and it is required that $\Phi^-(\infty)=0$. A solution of this problem is given by Gahov in his recent book [*Kraevye zadači*, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958; MR **21** #2879]. The same problem but with the weaker restriction $g(t) \in L_p(C)$, $p>1$, was treated by B. V. Hvedelidze [Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **23** (1956), 3–158; MR **21** #5873]. The author obtains Hvedelidze's results imposing not only his weaker restriction on $g(t)$ but also mere continuity on $G(t)$. (For a further generalization see the review below.)

J. F. Heyda (Cincinnati, Ohio)

3806:

Mandžavidze, G. F.; Hvedelidze, B. V. On the Riemann-Privalov problem with continuous coefficients. Dokl. Akad. Nauk SSSR **123** (1958), 791–794. (Russian)

This paper, written independently of that reviewed above, is a generalization of the topic treated there in that $G(t)$ is now a non-singular square matrix of order n with elements which are continuous functions on C and $g(t)$ is a matrix also of order n with elements belonging to $L_p(C)$, $p>1$. The problem of determining a sectionally-holomorphic matrix $\Phi(z) \in L_p(C)$, $p>1$, with elements representable as integrals of Cauchy type with specified principal parts at infinity, so that on C

$$(1) \qquad \Phi^+(t) = G(t)\Phi^-(t)+g(t),$$

is solved. Heretofore the elements of $G(t)$ were assumed to satisfy a Hölder condition on C [N. I. Mushelišvili, *Singulyarnye integral'nye uravneniya*, OGIZ, Moscow-Leningrad, 1946; translated by J. R. M. Radok as *Singular integral equations*, Noordhoff, Groningen, 1953; MR **8**, 586; **15**, 434].

J. F. Heyda (Cincinnati, Ohio)

3807:

Wolska-Bochenek, J. Sur un problème généralisé de Vécoua. Ann. Polon. Math. **7** (1960), 209–221.

Suppose L is a simple closed curve which bounds a domain S in the plane. Suppose $a_j(t)$, $f(t)$, $h_j(t, \tau)$ are defined for t and τ on L, and F is a real function. If these functions, as well as L, satisfy certain Hölder conditions, the author demonstrates the existence of a holomorphic function Φ in S whose derivatives satisfy the boundary condition

$$\operatorname{Re} \sum_{j=0}^{m} \{a_j(t)\Phi^{(j)}(t) + \int_L h_j(t, \tau)\Phi^{(j)}(\tau)\, ds\} = f(t)+F[t, \Phi(t), \Phi'(t), \cdots, \Phi^{(m)}(t)]$$

where ds is the element of arc length of L at the point τ. The problem leads to the study of certain singular integral equations. The case $F=0$ had previously been considered by Vekua [Soobšč. Akad. Nauk Gruzin. SSSR **2** (1941), 477–484; MR **6**, 123].

W. Rudin (Madison, Wis.)

3808:

Nikolaeva, G. A. On approximate construction of a conformal mapping by the method of conjugate trigonometric series. Trudy Mat. Inst. Steklov. **53** (1959), 236–266. (Russian)

The article contains details of results summarized in a previous publication [Dokl. Akad. Nauk SSSR **110** (1956), 180–183; MR **18**, 385]. A method of L. V. Kantorovič [cf. L. V. Kantorovič and V. I. Krylov, *Približennye metody vysšego analiza*, 3d ed., Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950; German translation, VEB Deutscher Verlag der Wissenschaften, Berlin, 1956; MR **13**, 77; **18**, 32] and a variation of this method are described. Some examples are numerically computed (partly on an electronic computer) and different methods compared. A new investigation of the problem of convergence and the estimation of the exactness of the process is made by considering the solution of a certain non-linear functional equation.

B. A. Amirà (Jerusalem)

3809:

Černin, K. E. Conformal mapping of regions consisting of rectangles onto the unit circle. Trudy Mat. Inst. Steklov. **53** (1959), 307–312. (Russian)

The author shows on a specific example the usefulness of his book [L. V. Kantorovič, V. I. Krylov and K. E. Černin, *Tablicy dlya čislennogo rešeniya graničnyh zadač teorii garmoničeskih funkcii*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956; MR **19**, 887]. Let $F(z)$ be the function which maps conformally a region R with boundary Γ on the unit-circle, the origin being the image of a given point $z_0 \in R$. Then

$$F(z) = (z-z_0)e^{-(g+ih)},$$

where $g=g(x, y)$ is the solution of the Dirichlet problem for the region R (with suitable boundary conditions) and $h=h(x, y)$ is the conjugate of $g(x, y)$. In some cases the function $h(x, y)$ is easy to find by solving the Neumann problem in R. If R is composed of a number of rectangles the computing of g and h is relatively simple, when using the book referred to.

B. A. Amirà (Jerusalem)

3810:

Gál, István S. Conformally invariant metrics on Riemann surfaces. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 1629–1633.

Let X be an arbitrary Riemann surface and let x and y

be two points of X. Let S_{xy} be a simply connected region of X containing x and y and such that the boundary of S_{xy} is a Jordan curve or a Jordan arc. We let Γ_{xy}^1 be the family of all possible boundary curves and Γ_{xy}^2 be the subfamily of Jordan curves. If λ_{xy}^i is the extremal length of the family Γ_{xy}^i $(i=1, 2)$, then $d^i(x, y) = \sqrt{\lambda_{xy}^i}$ is shown to be a metric on X save for the following exceptional cases. If X is conformally equivalent to the whole Riemann sphere or finite plane, then d^1 and d^2 are both identically zero. If X is equivalent to the unit disk, d^1 is identically zero. In all other cases, d^1 and d^2 are compatible with the locally euclidean topology of X. The author states several applications of these metrics, the first being another proof that the conformal mapping of one multiply connected Jordan region onto another such can be extended to be a homeomorphism of their closures. He also compares the uniform structures $\mathscr{U}^i$ generated by d^i $(i=1, 2)$, with the uniform structure $\mathscr{U}$ which agrees with the topology of the Riemann sphere. For example, he states that if X is a multiply connected Jordan domain, $\mathscr{U}^1 = \mathscr{U}$, and if X is planar and not simply connected, then $\mathscr{U}^2$ is complete and is strictly finer than $\mathscr{U}$.

G. Springer (Lawrence, Kans.)

3811:

Constantinescu, Corneliu. Sur le comportement d'une fonction analytique à la frontière idéale d'une surface de Riemann. C. R. Acad. Sci. Paris **245** (1957), 1995–1997.

Let γ be the relative boundary of a subregion R of an open Riemann surface and suppose that γ consists of a finite number of disjoint analytic Jordan curves. Denote by K the family of harmonic functions $u(z)$ on $\bar{R}$ such that $\int_\lambda du^* = 0$ along every dividing cycle λ on R. For the subclasses KY $(Y=B, D)$ of K, the author introduces the classes O_{KY} of regions R characterized by the following condition: every $u(z) \in KY$ has the property $u = L_0 u$, where L_0 is the normal linear operator minimizing the Dirichlet integral [Sario, Trans. Amer. Math. Soc. **72** (1952), 281–295; MR **13**, 735].

It is known that if p is a Kerékjártó-Stoïlow component of the ideal boundary of R, then p has vanishing capacity if and only if the Ahlfors-Beurling extremal length of the family of finite sets of disjoint analytic Jordan curves separating γ from p vanishes [M. Jurchescu, Thesis, Bucharest, 1956]. The author shows that if p has this property and if $R \in O_{KY}$ $(Y=B, D)$, then every single-valued analytic function $w(z)$ with the corresponding property Y on R tends to a limit as z approaches p.

L. Sario (Los Angeles, Calif.)

3812:

Teleman, C. Sur les structures homographiques d'une surface de Riemann. Rev. Math. Pures Appl. **4** (1959), 295–303.

By a homographic structure of order p on a Riemann surface R is meant a set of $p+1$ linearly independent (multivalued) meromorphic functions with the property that under continuation around a closed curve the final values of the functions are related to their original values by means of a fixed projective transformation.

The author shows that the homographic structures of order 1 are in one-to-one correspondence with the regular quadratic differentials on R, and analogous results are obtained for homographic structures of higher order.

H. L. Royden (Stanford, Calif.)

3813:

Nakai, Mitsuru. Purely algebraic characterization of quasiconformality. Proc. Japan Acad. **35** (1959), 440–443.

On a Riemann surface R, let $M(R)$ be Royden's algebra of complex-valued bounded and Tonelli absolutely continuous functions with the norm

$$\|f\| = \sup|f| + \left(\iint_R df \wedge {}^*d\bar{f}\right)^{1/2}.$$

The author proves that R and R' are quasiconformally equivalent if and only if $M(R)$ and $M(R')$ are algebraically isomorphic. They are conformally equivalent if and only if the algebras are isometric.

L. Ahlfors (Cambridge, Mass.)

3814:

Singh, S. K. Sur quelques applications des ordres approchés. Bull. Sci. Math. (2) **84** (1960), 11–16.

Let $f(z)$ be an entire function of finite order ρ, and let $n(r)$ denote the number of zeros of $f(z)$ in $|z| \leq r$, $n(r)=0$ for $r<1$, and $N(r) = \int_0^r \{n(t)/t\}dt$. The author gives new proofs, depending on proximate orders, of known results due to the reviewer [J. Indian Math. Soc. (N.S.) **5** (1941), 189–191; Math. Student **10** (1942), 80–82; **12** (1945), 67–70; MR **4**, 6, 137; **6**, 263] and R. P. Boas [Rend. Circ. Mat. Palermo (2) **1** (1952), 323–331; MR **14**, 1074]: if

$$\lambda_1 = \liminf_{r\to\infty} \{\log^+ n(r)/\log r\} > 0,$$

then

$$\liminf_{r\to\infty} n(r)/N(r) \leq \lambda_1,$$

$$\limsup_{r\to\infty} n(r)/N(r) \geq \limsup_{r\to\infty} \{\log^+ n(r)/\log r\};$$

and if $\rho<1$, $\limsup_{r\to\infty} N(r)/\log M(r) \geq 1-\rho$. It is also proved that if $\rho>0$, then

$$\liminf_{r\to\infty} \log M(r)/T(r) \leq \{(k+1)/(k-1)\}k^\rho$$

where $k = \{1+(1+\rho^2)^{1/2}\}/\rho$. *S. M. Shah* (Lawrence, Kans.)

3815:

Montel, Paul. Sur les propriétés tangentielles des fonctions analytiques. Rev. Math. Pures Appl. **3** (1958), 5–8.

The theorem of Picard on exceptional values of entire functions is described in terms of projective geometry and dual forms considered. If the equation in z

$$U - f(z) = f'(z)(Z-z)$$

has no solutions for three pairs (Z, U) corresponding to three points in line, then the function $f(z)$ meromorphic in the whole plane is of the form $Cz + C^1$. If $f(z)$ is regular for $|z|<1$ and the three exceptional points (Z, U) are $(0, 0)$, $(0, 1)$ and $(0, \infty)$, then (cf. Schottky's theorem)

$$|f(z)| \leq |a_0| + |a_1|\theta^2 + \Omega(a_0, \theta)(1-\theta)^{-2},$$

for $|z| \leq \theta^2$, where $f(z) = a_0 + a_1 z + \cdots$. There is a similar generalization of Landau's theorem.

Some "mixed" hypotheses are dismissed easily or connected with previous work.

A. J. Macintyre (Cincinnati, Ohio)

3816:

Shah, S. M.; Singh, S. K. Meromorphic functions with maximum defect sum. Tôhoku Math. J. (2) **11** (1959), 447–452.

The authors prove the following theorem. Let $f(z)$ be a meromorphic function of finite order ρ, such that some value has deficiency one (in the sense of Nevanlinna's theory) and such that the sum of all deficiencies is equal to two. Then ρ is a positive integer and the growth of $f(z)$ is regular.

The authors base their proof on the work of Edrei and Fuchs [Trans. Amer. Math. Soc. **93** (1959), 292–328; MR **22** #770]. In fact their theorem follows by a simple linear transformation from theorem 7 of the above paper cited by the authors. *A. Edrei* (Syracuse, N.Y.)

3817:

Shankar, Hari. Note on a theorem of Shah. Rend. Circ. Mat. Palermo (2) **8** (1959), 225–227.

Let $w(z)$ be a meromorphic function and let $n(r, a)$ and $N(r, a)$ have their usual meanings. Following the argument of the reviewer [J. Indian Math. Soc. (N.S.) **5** (1941), 189–191; MR **4**, 6], the author proves that

$$\liminf_{r\to\infty}\{\sum_1^q n(r, a_\nu)\}/\mathfrak{N}(r, a_i) \leq q\lambda_1(a_i),$$

where

$$\mathfrak{N}(r, a_i) = \max\{N(r, a_1), \cdots, N(r, a_q)\},$$

$$\lambda_1(a_i) = \liminf_{r\to\infty}\{\log \mathfrak{N}(r, a_i)\}/\log r.$$

S. M. Shah (Lawrence, Kans.)

3818:

Flett, Thomas Muirhead. A note on a maximal function. Tôhoku Math. J. (2) **12** (1960), 34–46.

E. Stein [Ann. of Math. (2) **68** (1958), 584–603; MR **20** #6630] has introduced the function

$$M_\lambda(\theta) = \sup_{0\leq\rho<1}\left\{\delta^{\lambda-1}\int_{\delta\leq|t|\leq\pi}\frac{|u(\rho, \theta+t)|^2}{|t|^\lambda}dt\right\}^{1/2}, \quad \delta = 1-\rho,$$

where $u(\rho, \theta)$ is the Poisson integral of an integrable f, and proved very useful maximal theorems. The author considers the function

$$L_{k,\lambda}(w, \theta) = \sup_{0\leq\rho<1}\left\{\delta^{\lambda-1}\int_{-\pi}^{\pi}\frac{w^k(\rho, \theta+t)}{|1-\rho e^{it}|^\lambda}dt\right\}^{1/k}, \quad \delta = 1-\rho,$$

for a function $w(\rho, \theta)$ non-negative and subharmonic in the circle $\rho<1$, and proves results which contain Stein's maximal theorems. The author follows the method used by Stein, but there are many simplifications. In particular the author gives a very direct proof of a theorem on the Cesàro means of power series using his own results for $L_{k,\lambda}(w, \theta)$. *G. Sunouchi* (Evanston, Ill.)

3819:

Zmorovič, V. A. Theory of special classes of univalent functions. I. Uspehi Mat. Nauk **14** (1959), no. 3 (87), 137–143. (Russian)

The author defines a plane schlicht δ-region B to be one such that any two distinct points of B can be joined by a δ-arc. A δ-arc, $\delta>0$, is a piecewise-smooth arc L of finite length with the property that the angle θ formed by the tangent to L at a non-corner point with a fixed direction of the chord of L satisfies $\sup\theta-\inf\theta<\delta$. Every subregion B' of B such that every two distinct points of B' can be joined by a δ-arc of B is called a δ-subregion. The author shows that if B_z is an arbitrary schlicht domain in the z-plane, and if $w=\phi(z)$ is a single-valued analytic function mapping B_z into an arbitrary schlicht domain B_w of the w-plane with a subdomain $B_z'\subset B_z$ mapping into $B'_w\subset B_w$, then if B'_w is a δ-subdomain of B_w, $0\leq\delta<\pi$, any analytic function $f(z)$ single-valued in B_z and satisfying

$$(1)\qquad \sup\arg\frac{f'(z)}{\phi'(z)}-\inf\arg\frac{f'(z)}{\phi'(z)} \leq \pi-\delta, \quad z\in B_z,$$

is schlicht in B_z'. This theorem is a generalization of a result for close-to-convex functions due to S. Ozaki [Sci. Rep. Tokyo Bunrika Daigaku A **2** (1935), 167–188] and W. Kaplan [Michigan Math. J. **1** (1952), 169–185; MR **14**, 966].

A simply-connected schlicht domain in the z-plane is called a region of class $B(\alpha;\beta)$, $0\leq\alpha\leq\pi$, $-\pi\leq\beta\leq0$, if it has the following property: if $z\in B(\alpha;\beta)$, $\operatorname{Im}(z)>0$, then the line segment with inclination α joining z to the real axis belongs entirely to $B(\alpha;\beta)$, and when $\operatorname{Im}(z)<0$ the line segment joining z to the real axis and making with the positive real axis the angle β also lies in $B(\alpha;\beta)$. If $\alpha-\beta=\pi$ the domain is convex in one direction [see M. S. Robertson, Amer. J. Math. **58** (1936), 465–472]. If $\alpha=-\beta$ the domain $B(\alpha;-\alpha)$ is symmetric with the real axis [see B. N. Rahmanov, Dokl. Akad. Nauk SSSR **91** (1953), 729–732; MR **15**, 413].

If $f(z)\in B(\alpha;\beta)$ and $f'(z)$ is regular in the closure of the domain defined by $|z|\leq1$, $|1-ze^{-i\gamma}|\geq\varepsilon$, $|1-ze^{-i\delta}|\geq\varepsilon$, $\gamma\leq\delta$, for sufficiently small $\varepsilon>0$, then $f(z)$ satisfies the equation

$$(2)\qquad Cf'(z) = \phi'(z)\left[1+\frac{\cos\sigma}{\pi}e^{-i\sigma}\int_0^{2\pi}\frac{ze^{-i\theta}}{1-ze^{-i\theta}}d\mu(\theta)\right],$$

where $z\phi'(z)=z(1-ze^{-i\gamma})^{\lambda-1}\cdot(1-ze^{-i\delta})^{-\lambda-1}$ is star-like in $|z|<1, \lambda=1-(\alpha-\beta)/\pi$, σ = real constant, $Cf'(0)=1$ and $\mu(\theta)$ is a non-decreasing function in $[0, 2\pi]$, $\mu(2\pi)-\mu(0)=2\pi$. By means of (2) it is shown that $B(\alpha;\beta)$ is contained in the class of close-to-convex functions.

M. S. Robertson (New Brunswick, N.J.)

3820:

Zmorovič, V. A. Theory of special classes of univalent functions. II. Uspehi Mat. Nauk **14** (1959), no. 4 (88), 169–172. (Russian)

In this sequel to the paper reviewed above the author shows that if $f(z)\in B(\frac12\pi;-\frac12\pi)$, $f(0)=0$ and if $v(t)$ is a non-decreasing function on $[-1, 1]$, then

$$F(z) = \int_{-1}^1\frac1t f(zt)\,dv(t)\in B(\tfrac12\pi;-\tfrac12\pi).$$

If also $\operatorname{Re} f'(i)=0$, then for the class of functions $F(z)$ the circle $|z|<1$ of univalency cannot be enlarged.

Let $G(z)$ be regular in $|z|<1$ and $\operatorname{Re} G(z)\geq0$ in $|z|\leq r$, $0<r\leq1$. Let $z_1=re^{i\alpha}$, $z_2=re^{i\beta}$ $(z_1\neq z_2)$ be points on $|z|=r$ such that $\operatorname{Re} G(z_1)=\operatorname{Re} G(z_2)=0$ and $\operatorname{Im} F(z_1)\cdot\operatorname{Im} F(z_2)\leq0$. Then for any function $\phi(z)$ convex in $|z|<r$ and

regular in $|z|<1$ and any non-decreasing function $\mu(\theta)$ in $[0, 2\pi]$ normalized so that $\mu(2\pi)-\mu(0)=2\pi$, the formula

$$f(z) = \int_0^{2\pi} \Phi(z, \theta) d\mu(\theta),$$

where $\Phi(z, \theta)=\int_0^z \phi'(\zeta)F(\zeta e^{i\theta})d\zeta$, defines a class of functions regular in $|z|<1$ and schlicht in $|z|<r$. This class is contained in the class of close-to-convex functions. The circle, $|z|<r$, of univalence for this class cannot be enlarged.

If $f(z)$ is regular and convex in $|z|<1$, then the function

$$F(z) = \lambda f(z)+\mu z f'(z)+C$$

where a, λ, u, C are complex numbers, with $\mu\neq 0$, $|\arg(\lambda/\mu)|<\frac{1}{2}\pi$, is close-to-convex in $|z|<1$.

M. S. Robertson (New Brunswick, N.J.)

3821:

Dunducenko, L. E. Deformation theorems for starlike functions. Acad. R. P. Romîne. An. Romîno-Soviet. Ser. Mat.-Fiz. (3) **13** (1959), no. 3 (30), 63–75. (Romanian)

The author applies a variational method of Zmorovič [Ukrain. Mat. Ž. **4** (1952), 276–298; MR **15**, 301] to solve extremal problems for some classes of p-valent starlike functions. First let $f(z)=z^p+a_2z^{p+1}+\cdots$, $p\geq 1$, be regular, p-valent and starlike in $|z|<1$. The author quotes from Zmorovič's Kiev thesis [1950] and deduces from a theorem of A. W. Goodman [Trans. Amer. Math. Soc. **68** (1950), 204–223; MR **11**, 508] the formula

$$f(z) = z^p \exp\{-\pi^{-1}\int_{-\pi}^{\pi} \log(1-ze^{-i\theta})d\mu(\theta)\},$$

with μ nondecreasing, $\mu(-\pi+0)=\mu(-\pi)=0$, $\int_{-\pi}^{\pi} d\mu(\theta)=2p\pi$. He deduces the exact bounds $r^p(1+r)^{-2p}\leq|f(z)|\leq r^p(1-r)^{-2p}$,

$$pr^{p-1}(1-r)(1+r)^{-2p-1} \leq |f'(z)| \leq pr^{p-1}(1+r)(1-r)^{-2p-1},$$

and some similar results. Next he considers functions that are starlike and univalent in $q<|z|<1$, normalized by $\int_{|z|=\rho} z^{-1}f(z)dz=2\pi i$, $q<\rho<1$. By using a structure formula for this class [Zmorovič, Mat. Sb. (N.S.) **32 (74)** (1953), 633–652; MR **14**, 1075] he determines exact bounds for $|f(z)|$ and $|f'(z)|$ in terms of theta functions.

R. P. Boas, Jr. (Evanston, Ill.)

3822:

Geronimus, Ya. L. Some estimates for the coefficients of bounded functions. Izv. Akad. Nauk SSSR Ser. Mat. **24** (1960), 203–212. (Russian)

Let $f(z)=\sum_{k=0}^{\infty} a_kz^k$ be regular and $|f(z)|<1$ in $|z|<1$. If $n>2m$, it is well known that $|\alpha_m|^2\leq 1-|\alpha_n|$. The author now considers the range $3m/2<n\leq 2m$ and proves that in this range

$$|\alpha_m| \leq \mu,\ \mu^2-\mu^3 = |\alpha_n|^2, \quad \text{if } |\alpha_n| \leq \alpha = 14\sqrt{3}-24,$$

$$\leq \frac{4\sqrt{3}}{9}\left\{1-\frac{9}{8}|\alpha_n|+\left(1-\frac{3}{4}|\alpha_n|\right)^{3/2}\right\}^{1/2}, \quad \text{if } |\alpha_n| \geq \alpha,$$

where $\mu>2/3$. This inequality is sharp and the extremal functions are certain rational functions that are explicitly given. The author also considers some related questions.

A. W. Goodman (Lexington, Ky.)

3823:

Barsoum, F. R.; Nassif, M. On the convergence of the product series of simple sets of polynomials in a general region. II. Nederl. Akad. Wetensch. Proc. Ser. A **63** = Indag. Math. **22** (1960), 333–337.

This paper completes part I [Nassif, same Proc. **60** (1957), 598–607; MR **20** #3295] by proving a best possible result for the region in which the product set of two sets of polynomials represents a function. As in part I, let C be a simple closed curve, and let $z=\psi(t)=ct^{-1}+\varphi(t)$ map the unit disk onto the exterior of C. Let $\chi(t)=(t/c)\psi(t)=\sum_{k=0}^{\infty} a_kt^k$, $\{\chi(t)\}^n=\sum_{k=0}^{\infty} a_k^{(n)}t^k$, $S_n(R)=\sum_{k=0}^{n} |a_k^{(n)}|R^k$, $S(R)=\lim\sup\{S_n(R)\}^{1/n}$. Define β as in part I. Then if two simple absolutely monic sets are effective in the closed interior of C, their product set will represent in this set every function that is regular in the closed interior of the level curve $C_{\beta/S(\beta)}$.

R. P. Boas, Jr. (Evanston, Ill.)

3824:

Guo, Zhu-rui. The improvement of S. N. Mergelyan's theorems. Acta Math. Sinica **9** (1959), 271–280. (Chinese. English summary)

Author's summary: "This paper extends S. N. Mergelyan's converse theorems of Tchebycheff approximation in the complex domain to that of the corresponding theorem of one real variable given by De la Vallée Poussin, and gives some corollaries."

F. C. Hsiang (Taipeh)

3825:

Sharma, A. Remark on a theorem of Cinquini. Acta Math. Acad. Sci. Hungar. **11** (1960), 93–96. (Russian summary, unbound insert)

The author establishes a mean value theorem for functions $f(z)$ analytic in a circle. The theorem relates the values $f(z_0)$, $f(z_1)$, $f'(\frac{1}{2}z_0+\frac{1}{2}z_1)$, $f'''(\xi)$, for appropriate choice of ξ. The result extends previous results of Cinquini [Rend. Circ. Mat. Palermo **61** (1937), 73–82] and Montel [J. Math. Pures Appl. (9) **16** (1937), 219–231].

W. Kaplan (Ann Arbor, Mich.)

3826:

Farrell, O. J. On approximation by non-vanishing functions. Proc. Amer. Math. Soc. **10** (1959), 888–890.

Let G be a simply connected region of the complex plane. Let S designate the set of functions which are analytic in the interior of G and which vanish either identically or not at all interior to G. $F(z)$ is any member of S which is uniformly limited. Let $f(z)$ be analytic, uniformly limited, and have k zeros interior to G. Set

$$M = \inf_F \sup_{z\in G} |F(z)-f(z)|.$$

The author derives an estimate for M and two theorems on the number of zeros in G of functions which approximate $f(z)$ closer than M.

P. J. Davis (Washington, D.C.)

3827:

Tsuboi, Teruo. Notes on Bergman representative domains and minimal domains. Sci. Rep. Saitama Univ. Ser. A **3** (1959), 49–64.

The Bergman kernel function for domains in the space of k complex variables and some related domain functions are formulated in a very convenient matrix notation.

Then theorems of the type proved by Maschler [Pacific J. Math. **6** (1956), 501–516; MR **18**, 473] are proved in somewhat different formulations. For example, there is given a necessary and sufficient condition that a representative domain of a domain D with respect to a fixed point $u \in D$ be simultaneously a minimal domain of D with respect to u. *G. Springer* (Lawrence, Kans.)

3828:

Hitotumatu, Sin. Some remarks on a quasi-pseudo-conformal mapping of Reinhardt circular domains. Comment. Math. Univ. St. Paul. **8** (1960), 1–21.

In the space of two complex variables, S. Bergman [J. Math. Mech. **7** (1958), 937–956; MR **21** #1399] showed that a QPC (quasi-pseudo-conformal) mapping from a hypersphere onto a Reinhardt circular domain gives a finite distortion of the Bergman's invariant metric under certain restrictions. The author remarks that a QPC mapping for a Reinhardt circular domain does not necessarily give a finite distortion of the invariant metric without any restriction. This fact is shown by direct computation in case of an ellipsoid of revolution as a special Reinhardt circular domain. It is also pointed out that the distortion is bounded from above and below if everything is restricted only on the real section.

Y. Komatu (Tokyo)

SPECIAL FUNCTIONS

3829:

Luke, Yudell L. On economic representations of transcendental functions. J. Math. and Phys. **38** (1959/60), 279–294.

Following a technique given by the reviewer [same J. **30** (1952), 226–231; MR **13**, 690], the author generates in a systematic fashion a large class of useful rational approximations for some of the basic transcendents of analysis. These include generalized hypergeometric series. The convergence problem is discussed and numerical results are given for the Whittaker functions. Reference to two previous papers may be found in ibid. **37** (1958), 110–127 [MR **20** #5558].

R. E. Bellman (Santa Monica, Calif.)

3830:

Sokolov, A. A.; Muradyan, R. M.; Arutyunyan, V. M. Development of the WKB approximation method. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1959**, no. 4, 61–78; no. 6, 64–86. (Russian)

The generalizations of the WKB method proposed by Kramers and Langer are utilized for more exact approximations of special functions of mathematical physics. To attain the latter the differential equation investigated is replaced by a specially selected equation, whose solution can be obtained in terms of simpler functions. Arbitrary constants of the selected equation are determined by equating it to the exact value of the investigated function at one of the most convenient points. In this way is obtained an asymptotic development of an investigated function valid in a whole interval of variation of the argument.

The results are used for the investigation of the functions represented by the non-degenerated hypergeometric Gauss' functions, as well as degenerated hypergeometric functions. It is also shown that by means of asymptotic solutions it is possible to determine the eigenvalues, and thus the energy levels, for the Schrödinger wave equation. *R. M. Evan-Iwanowski* (Syracuse, N.Y.)

3831:

Anastassiadis, Jean. Définitions fonctionnelles de la fonction B(x, y). Bull. Sci. Math. (2) **83** (1959), 24–32.

Two theorems are proved: (A) If $f(x)>0$, defined and decreasing for $x>0$, is such that $f(x+1)=(x/(x+y))f(x)$, $y>0$, with $f(1)=1/y$, then $f(x)$ is identical to the beta function $\mathrm{B}(x, y)$. (B) If $f(x, y)>0$, defined for $x>0$, $y>0$, is such that

$$f(x+1, y+1) = [xy/(x+y)(x+y+1)]f(x, y),$$

with $f(1, 1)=1$ and

$$g(x, y) = [(x+y)^{x+y}/x^x y^y]f(x, y)$$

is decreasing for $x>0$, $y>0$ (i.e., $x_1>x_2$ and $y_1>y_2$ imply $g(x_1, y_1)<g(x_2, y_2)$), then $f(x, y)$ is identical to the beta function $\mathrm{B}(x, y)$.

Theorem (A) is generalized.

A. E. Danese (Schenectady, N.Y.)

3832:

Jager, H. A note on the continued fraction of Gauss. Nederl. Akad. Wetensch. Proc. Ser. A **63** = Indag. Math. **22** (1960), 181–186.

The following theorem on "abstract hypergeometric series" [cf. A. van der Sluis, Thesis, Amsterdam, 1956; Canad. J. Math. **10** (1958), 592–612; MR **20** #2561] is proved. Let a function $[s, k]$ be defined for every $s \in S$ and $k \in I$, S denoting an arbitrary set and I the set of all integers, having its range in a commutative ring R with unit element e and satisfying the following three conditions:

$$(1)\qquad \begin{vmatrix} e & [a, k] & [a, k+h] \\ e & [b, l] & [b, l+h] \\ e & [c, m] & [c, m+h] \end{vmatrix} = 0$$

for all $a, b, c \in S$ and all $h, k, l \in I$; (2) $[\sigma, 0]=0$ for a certain $\sigma \in S$; (3) $[\sigma, m]^{-1}$ exists for each $m \neq 0$. In particular, here R may be the field of all complex numbers, S the set of all complex numbers, $\sigma=0$, $[s, k]=s+k$; or, R and S as stated, $\sigma=1$, $[s, k]=1-sq^k$ where q is a complex number different from a root of unity. Furthermore, let the function $[s, k]_h$, $s \in S$, $k \in I$, $h \in I$, $h \geqq 0$, be defined as follows: $[s, k]_0=e$; $[s, k]_h=[s, k][s, k+1]\cdots[s, k+h-1]$, $h>0$. The formal power series in the variable x with coefficients from R,

$$F([a, k], [b, l]; [c, m]; x) = \sum_{\mu=0}^{\infty} \frac{[a, k]_\mu [b, k]_\mu}{[c, m]_\mu [\sigma, 1]_\mu} x^\mu,$$

which is defined if $[c, m+\mu]$ has an inverse for all $\mu>0$, is called an abstract hypergeometric series. Let $[c, m+\mu]$ have an inverse in R for every $\mu=0, 1, 2, \cdots$. Then

$$(\mathrm{a})\qquad \frac{F([a, k], [b, l]; [c, m]; x)}{F([a, k], [b, l+1]; [c, m+1]; x)} = e + \frac{a_1 x}{e} + \frac{a_2 x}{e} + \cdots + \frac{a_n x}{e} + \cdots$$

with

$$a_{2v} = -\frac{[b, l+v]\{[c, m+2v-1]-[a, k+v-1]\}}{[c, m+2v-1][c, m+2v]}$$
$$(v = 1, 2, \cdots),$$

$$a_{2v+1} = -\frac{[a, k+v]\{[c, m+2v]-[b, l+v]\}}{[c, m+2v][c, m+2v+1]}$$
$$(v = 0, 1, 2, \cdots).$$

For any $n=0, 1, 2, \cdots$ the coefficient of x^{n+1} in the power series given by (a) is different from the corresponding coefficient in the power series for the terminating continued fraction $e+\frac{a_1x}{e}\, + \cdots + \frac{a_nx}{e}$ if $\prod_{v=1}^n a_v \neq 0$. If $k=l=m=0$ and $[s, k]=s+k$, $\sigma=0$, the continued fraction of Gauss is obtained; if $k=l=m=0$ and $\sigma=1$, $[s, k]=1-sq^k$, the continued fraction of Heine is obtained.

E. Frank (Chicago, Ill.)

3833:

Berg, Lothar. Über eine spezielle Folge von Polynomen. Math. Nachr. **20** (1959), 152–158.

The author establishes various properties of the polynomials $P_n(s)$ defined by the generating function $e^{-sx}/(1-x)^s=\sum_{n=0}^\infty P_n(s)x^n$. Among these properties are a recursion formula, the asymptotic relations $\Gamma(s)P_n(s)\sim e^{-s}n^{s-1}$ $(n\to\infty)$ and $P_n(-k)\sim(-1)^k k^{n-k}/(n-k)!$ $(k\to\infty$, k a positive integer), and a connection with generalized Laguerre polynomials. The equation

$$\int_0^\infty e^{-t}t^{s-1}g(t)dt = \Gamma(s)\sum_{n=0}^\infty P_n(s)g^{(n)}(s)$$

holds for a certain class of Laplace transforms g.

R. R. Goldberg (Evanston, Ill.)

ORDINARY DIFFERENTIAL EQUATIONS

See also 3887.

3834:

Ionescu, D. V. L'intégration d'une équation différentielle. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. **8** (1957), 275–289. (Romanian. Russian and French summaries)

The differential equation in question has the form $\Delta_n(y)=0$, where $\Delta_n(y)$ is the Wronskian of the functions $y, y', \cdots, y^{(n)}$. The case $n=1$ is trivial, the case $n=2$ has been studied by Darboux. In the general case the solution can be established in terms of elementary functions. The more general equation $\Delta_n(y)=Ae^{ax}$ is also considered.

G. Szegő (Stanford, Calif.)

3835:

Perčinkova-V'čkova, Danica. Sur une application du procédé de Gauss-Chiò pour condensation des déterminants. Bull. Soc. Math. Phys. Macédoine **8** (1957), 19–21. (Macedonian. French summary)

Du résumé de l'auteur: "On montre que la propriété

$$W(y_1, y_2, \cdots, y_n) = y_1^n W\left(1, \frac{y_2}{y_1}, \frac{y_3}{y_1}, \cdots, \frac{y_n}{y_1}\right),$$

(W Wronskien) peut être démontrée à l'aide du procédé de Gauss-Chiò pour la condensation des déterminants, lequel on utilise d'ordinaire pour l'évaluation des déterminants à éléments numériques."

3836:

Opial, Z. Sur l'équation différentielle ordinaire du premier ordre dont le second membre satisfait aux conditions de Carathéodory. Ann. Polon. Math. **8** (1960), 23–28.

A simple proof of a theorem of G. Scorza Dragoni [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12** (1952), 55–61; MR **13**, 831] with applications to a differential inequality.

G.-C. Rota (Cambridge, Mass.)

3837:

Corduneanu, C. Sur une classe de systèmes différentiels non-linéaires. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I (N.S.) **3** (1957), 31–36. (Russian and Romanian summaries)

The author [Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **2** (**50**) (1958), 397–400] has recently extended a result of A. Bielecki [Bull. Acad. Polon. Cl. III **4** (1956), 261–264, 265–268; MR **18**, 494] for a single ordinary differential equation $dy/dx=f(x, y)$, $y(x_0)=y_0$. In the present note, again using the Banach-Cacciopoli-Schauder-Tychonoff fixed point method, a further extension is made to systems: $dy_i/dx=f_i(x, y_1, \cdots, y_n)$ $(i=1, 2, \cdots, n)$, where the real-valued functions f_i are such that there exists a function $\lambda(x)$, continuous and non-negative on $0\leq x<a$, and two numbers $\alpha>1$ and $k\geq 0$, such that

$$|f_i(x, y_1, \cdots, \bar{y}_i, \cdots, y_n)-f_i(x, y_1, \cdots, \bar{\bar{y}}_i, \cdots, y_n)| \leq \lambda(x)|\bar{y}_i-\bar{\bar{y}}_i|,$$

$$|f_i(x, y_1, \cdots, y_{i-1}, 0, y_{i+1}, \cdots, y_n)| \leq k\lambda(x)\exp\left(\alpha\int_0^x \lambda(t)dt\right)$$

for $i=1, \cdots, n$; $0\leq x<a$, $-\infty<y_i$, $\bar{y}_i$, $\bar{\bar{y}}_i<+\infty$. It is shown that, for any given y_i^0, there exists at least one solution $y_i(x)$ of the system, on $0\leq x<a$, such that $y_i(0)=y_i^0$; and that, further, $y_i(x)$ satisfies the inequality

$$|y_i(x)| \leq \frac{\alpha}{\alpha-1}\left(|y_i^0|+\frac{k}{a}\right)\exp\left(\alpha\int_0^x\lambda(t)dt\right)$$

on $0\leq x<a$.

J. B. Diaz (College Park, Md.)

3838:

Corduneanu, C. Quelques remarques concernant certaines classes de systèmes différentiels. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I (N.S.) **3** (1957), 37–44. (Russian and Romanian summaries)

The paper contains extensions, to systems of ordinary differential equations, of previous results for a single equation [C. Corduneanu, same An. **2** (1956), 33–52; C. R. Acad. Sci. Paris **245** (1957), 21–24; MR **20** #3344; **19**, 652].

Sharper results are also obtained in the case of a single equation. For example, suppose that $f(x, y)$, $\partial f(x, y)/\partial y$ are real valued and continuous, with $|f(x, y)|$ bounded, and that there exists a number $m>0$ such that $\partial f/\partial y\leq -m<0$, all on $x\geq 0$, $-\infty<y<+\infty$. Then there exists

exactly one solution $y(x)$ of the initial value problem $dy/dx = f(x, y)$, $x \geqq 0$; $y(0) = y_0$, which is bounded in absolute value on $x \geqq 0$. *J. B. Diaz* (College Park, Md.)

3839:

Filippov, A. F. On continuous dependence of a solution on the initial conditions. Uspehi Mat. Nauk **14** (1959), no. 6 (90), 197–201. (Russian)

It is well known that for a differential system $dx/dt = f(t, x)$, $x = (x_1, \cdots, x_n)$, f continuous in (t, x), existence and uniqueness of the solution $x(t)$, $t \in I = [0, T]$, $T > 0$, for all initial values $a = x(0)$ of a domain $A \subset E_n$, implies the continuous dependence of $x(t)$ on the initial values a, $a \in A$. In the present paper no continuity requirement is made on f. For $n > 1$, the following set of conditions is proved to be necessary and sufficient for the continuous dependence of $x(t)$ on a, $a \in A$: (1) for any $a \in A$, if $a = x_1(0) = x_2(0)$, then $x_1(t)$ and $x_2(t)$ coincide in any subinterval of I where they both exist; (2) for any $a \in A$ there is an $\varepsilon > 0$ such that if $a_n \to a$, $a_n \in A$, then the corresponding solutions $x_n(t)$, $x_n(0) = a_n$, exist in $[0, \varepsilon]$ and are equicontinuous in $[0, \varepsilon]$; (3) for any subinterval $I_1 = [0, \varepsilon] \subset I$, the limit of any uniformly convergent sequence $x_n(t)$, $t \in I_1$, of solutions with initial values $a_n \to a$, a_n, $a \in A$, is itself the solution $x(t)$ with initial values a. Sufficient conditions are given and examples are discussed. For $n = 1$ see A. F. Andreev and Yu. S. Bogdanov, Uspehi Mat. Nauk **13** (1958), no. 3 (81), 165–166 [MR **20** #4070]. *L. Cesari* (Ann Arbor, Mich.)

3840:

Mizuno, Hirobumi. Sur les équations différentielles algébriques du type $f(y, y') = 0$. Proc. Japan Acad. **36** (1960), 55–58.

The differential equation $f(y, y') = 0$ is considered, where $f(Y, Z)$ is a polynomial and the curve $f(Y, Z) = 0$ is of arbitrary genus p. In the neighbourhood of any simple point (y_0, z_0) of this curve there exists a local solution $\psi(x)$ of the differential equation with $\psi(x_0) = y_0$, $\psi'(x_0) = z_0$, and by analytic continuation a global solution can be obtained. The classic result of Briot and Bouquet is that unless $p = 0$ or 1 this solution will not, in general, be uniform.

In this paper the author shows that in the general case there does, however, exist an infinitely multiform solution, and gives an explicit formula for it.

F. M. Arscott (Battersea, London)

3841:

Berezovs'kiĭ, M. I. On the transformation of differential operators. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 819–823. (Ukrainian. Russian and English summaries)

Author's summary: "This note gives necessary and sufficient conditions which determine the main part (in Laurent's sense) of the kernel of the contour-integral transformation $g(x) = f(x) + \int_C K(x, w) f(w) dw$ taking the differential operator $M = d^n/dw^n$ into a differential operator

$$\frac{d^n}{dx^n} + p_{n-2}(x) \frac{d^{n-2}}{dx^{n-2}} + p_{n-3}(x) \frac{d^{n-3}}{dx^{n-3}} + \cdots + p_1(x) \frac{d}{dx} + p_0(x)$$

with continuous coefficients."

3842:

Harasahal, V. H. A generalization of a theorem of Bohr and Neugebauer. Prikl. Mat. Meh. **23** (1959), 595 (Russian); translated as J. Appl. Math. Mech. **23**, 844–845.

The author shows for a linear system $\dot{x} = A(t)x + b(t)$ with $A(t)$ periodic and $b(t)$ almost periodic that every bounded solution is almost periodic.

H. A. Antosiewicz (Los Angeles, Calif.)

3843:

Halanay, A. Comportement asymptotique des solutions des équations du second ordre dans le cas de la non-oscillation. Com. Acad. R. P. Romîne **9** (1959), 1121–1128. (Romanian. Russian and French summaries)

Author's summary: "Soit l'équation $\ddot{x} + f(t)x = 0$ dans le cas de la non-oscillation. Alors elle admet une solution principale u_1, telle que $\int_a^\infty 1/u_1^2\, dt = \infty$ et $u_2 = u_1 \int_a^t 1/u_1^2(s)\, ds$ soit une solution linéairement indépendante de u_1 et telle que $\lim_{t\to\infty} u_2/u_1 = \infty$. Théorème 1: Si $\int_a^\infty u_1 u_2 (f-g)\, dt < \infty$, l'équation $\ddot{x} + g(t)x = 0$ admet deux solutions telles que $v_1 = u_1(1+\varepsilon_1)$, $v_2 = u_2(1+\varepsilon_2)$, où

$$\lim_{t\to\infty} \varepsilon_1 = \lim_{t\to\infty} \varepsilon_2 = 0; \quad \dot{v}_1 = \dot{u}_1(1+\tilde{\varepsilon}_1) + \frac{\hat{\varepsilon}_1}{u_2},$$

$$\dot{v}_2 = \dot{u}_2(1+\tilde{\varepsilon}_2) + \frac{\hat{\varepsilon}_2}{u_1},$$

$$\lim_{t\to\infty} \tilde{\varepsilon}_1 = \lim_{t\to\infty} \hat{\varepsilon}_1 = \lim_{t\to\infty} \tilde{\varepsilon}_2 = \lim_{t\to\infty} \hat{\varepsilon}_2 = 0.$$

Théorème 2: Si $\int_a^\infty u_2^2 (f-g) dt < \infty$, l'équation $\ddot{x} + g(t)x = 0$ admet deux solutions telles que $v_1 = u_1(1+\varepsilon_1)$, $v_2 = u_2(1+\varepsilon_2)$ où $\varepsilon_1 = o(u_1/u_2)$, $\varepsilon_2 = o(u_1/u_2)$, $\dot{v}_1 = \dot{u}_1(1+\tilde{\varepsilon}_1) + \hat{\varepsilon}_1/u_2$, $\dot{v}_2 = \dot{u}_2(1+\tilde{\varepsilon}_2) + \hat{\varepsilon}_2 \dot{u}_1$, $\tilde{\varepsilon}_1 = o(u_1/u_2)$, $\hat{\varepsilon}_1 = o(u_1/u_2)$, $\tilde{\varepsilon}_2 = o(u_1/u_2)$, $\hat{\varepsilon}_2 = o(1)$. Théorème 3: Si $u_1 \dot{u}_2$ est bornée pour $t \geqq t_0$, $g - f$ ne change pas de signe pour $t \geqq t_0$, et si l'équation $\ddot{x} + g(t)x = 0$ admet une solution telle que $v_1 = u_1(1+\varepsilon_1)$, $\dot{v}_1 = \dot{u}_1(1+\tilde{\varepsilon}_1) + \hat{\varepsilon}_1/u_1$, alors $\int_a^\infty u_1 u_2 (f-g) ds$ converge. Théorème 4: Si $u_1 u_2$ est bornée pour $t \geqq t_0$, $g - f$ ne change pas de signe pour $t \geqq t_0$ et si l'équation a une solution telle que $v_2 = u_2(1+\varepsilon_2)$, $\dot{v}_2 = \dot{u}_2(1+\tilde{\varepsilon}_2) + \hat{\varepsilon}_2 \dot{u}_1$, $\varepsilon_2 = o(u_1/u_2)$, $\tilde{\varepsilon}_2 = o(u_1/u_2)$, $\hat{\varepsilon}_2 = o(1)$, alors $\int_a^\infty u_2^2 (f-g) ds$ est convergente." *R. R. D. Kemp* (Kingston, Ont.)

3844:

Brinck, Inge. Self-adjointness and spectra of Sturm-Liouville operators. Math. Scand. **7** (1959), 219–239.

Let $q(x)$ be real valued and locally integrable on $(-\infty, \infty)$. Many known theorems concerning (1) $Lu \equiv -u'' + q(x)u$ involve a condition (2) $-q(x) \leqq C$ or an upper limitation for $\int^x \max(0, -q(t)) dt$. The author replaces these by variants of (3) $\int_J q(x) dx \geqq -C$ for all intervals J of length $\leqq 1$. His main tool is a simple estimate for Riemann-Stieltjes integrals due to Ganelius [C. R. Acad. Sci. Paris **242** (1956), 719–721; MR **17**, 609]. For example, it is shown that if $u \in L^2(-\infty, \infty)$ is a solution of $Lu = 0$, then (3) implies that $u' \in L^2(-\infty, \infty)$, so that (1) is of the limit point type at $x = \pm\infty$ and L (with a suitably defined domain) is self-adjoint. Also $L + 2C^2 I \geqq 0$. Furthermore, L has a discrete spectrum if and only if $\int_a^{a+h} q dx \to \infty$ as $|a| \to \infty$ for every $h > 0$. Finally, the author gives a characterization of the domain of $[L + (2C^2+1)I]^{1/2}$; in this connection, cf. results of Heinz [Math. Ann. **135** (1958), 1–49; MR **21** #743] on real, non-negative self-adjoint, ordinary differential operators of arbitrary even order. *P. Hartman* (Baltimore, Md.)

3845:

Nehari, Zeev. Some eigenvalue estimates. J. Analyse Math. 7 (1959), 79–88.

The main result is the following lemma: Let $K(x, t)$ be a real, symmetric, square-integrable, positive-definite kernel on $a \leq x, t \leq b$. Let C be the set of non-negative, monotonic functions $g(x)$ of $L^2(a, b)$. Let $\lambda = \lambda(g)$ be the lowest eigenvalue for $u(x) = \lambda \int_a^b K(x, t)g(x)g(t)u(t)dt$ and let $c = \inf_C \lambda^{1/2}(g) \int_a^b g(x)dx$. Then $c = \inf_H \lambda^{1/2}(h) \int_a^b h(x)dx$ where H is the set of non-negative step-functions $h(x)$ on $[a, b]$ with at most one discontinuity. {The proof shows that C can be any cone of non-negative functions in $L^2(a, b)$ and H any subset of C such that the smallest cone containing H is dense in C.} This lemma is used to show that if $p(x) \geq 0$ is monotone and of class $L^1(a, b)$, then the lowest eigenvalue for (1) $y'' + \lambda p(x)y = 0$, $y(a) = y(b) = 0$ satisfies $\lambda^{1/2} \int_a^b p^{1/2}(x)dx > \frac{1}{2}\pi$. The constant $\frac{1}{2}\pi$ is the best possible. Other applications give lower estimates for the nth eigenvalue for (1) and for the lowest eigenvalue for $y^{(4)} - \lambda p(x)y = 0$, $y(a) = y(b) = y'(a) = y'(b) = 0$.

P. Hartman (Baltimore, Md.)

3846:

Lučina, A. A. Qualitative analysis of a non-linear differential equation of the second order of a self-oscillating system with bounded increment region. Radiotehn. i Èlektron. 4 (1959), 440–448 (Russian); translated as Radio Engrg. and Electronics 4, no. 3, 127–139.

Author's summary: "The nature of the solutions and the type of the limit cycle of a self-oscillation equation are investigated for the case of large non-linearity, by constructing the phase portrait of the system. The small-parameter method [Tihonov, Mat. Sb. (N.S.) **22 (64)** (1948), 193–204; MR **9**, 588] is used."

3847a:

Magiros, Dem. G. Subharmonics of any order in case of non-linear restoring force. I. Prakt. Akad. Athēnōn **32** (1957), 77–85. (Greek summary)

3847b:

Magiros, Dem. G. Subharmonics of order one third in the case of cubic restoring force. II. Prakt. Akad. Athēnōn **32** (1957), 101–108. (Greek summary)

3847c:

Magiros, Dem. G. Remarks on a problem of subharmonics. Prakt. Akad. Athēnōn **32** (1957), 143–146. (Greek summary)

3847d:

Magiros, Dem. G. On the singularities of a system of differential equations, where the time figures explicitly. Prakt. Akad. Athēnōn **32** (1957), 448–451. (Greek summary)

An abridged version of results later published elsewhere [Information and Control **1** (1958), 198–227; MR **20** #5325].

J. L. Massera (Montevideo)

3848:

Zubov, V. I. On almost periodic solutions of systems of differential equations. Vestnik Leningrad. Univ. **15** (1960), no. 1, 104–106. (Russian. English summary)

Consider an n-vector system (1) $\dot{x} = f(x; t)$ such that f is (a) continuous in $E_n \times \{t \in (-\infty, +\infty)\}$ and uniformly as to t in any compact region of E_n; (b) Lipschitzian in x with fixed constant; (c) almost periodic in t for x fixed. The system is said to have the convergency property (c.p.) if it has a unique almost periodic solution which is asymptotically stable and uniformly so as to initial time t_0. Necessary and sufficient conditions for c.p. are: (a) any solution $x(t; x_0; t_0)$ is bounded for $t \geq t_0$; (b) given $\varepsilon, r > 0$ there corresponds $\delta(\varepsilon, r) > 0$ such that if $\|x_0 - y_0\| < \delta$ then $\|x(t; x_0; t_0) - x(t; y_0; t_0)\| < \varepsilon$ and $\to 0$ uniformly in t_0 as $t \to +\infty (|x_0| < r, |y_0| < r)$; (c) corresponding to $0 < \varepsilon$ and any x_0 there are numbers l, T such that in any interval $(\alpha, \alpha + l)$ there can be found a τ such that $\|x(t+\tau; x_0; t_0) - x(t; x_0; t_0)\| < \varepsilon$ for $t > t_0 + T$. Another set of sufficient conditions in terms of a complicated Liapunov function is also given, with complementary details when (1) has the form $\dot{x} = A(t)x + h(x; t) + p(t)$, where the matrix $A(t)$ and the vector $p(t)$ are almost periodic, $h(0; t) = 0$, and $\|h(x; t) - h(y; t)\| < B(t) \cdot \|x - y\|$, the matrix $B(t)$ being continuous and its terms non-negative. [Reference: Lyaščenko, Dokl. Akad. Nauk SSSR **104** (1955), 177–179; MR **17**, 616.]

S. Lefschetz (Princeton, N.J.)

3849a:

Corduneanu, C. Sur la stabilité asymptotique. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I (N.S.) **5** (1959), 37–40. (Russian and Romanian summaries)

3849b:

Corduneanu, C. Sur la stabilité asymptotique. II. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **10** (1959), 209–213. (Romanian. Russian and French summaries)

Let $x' = f(t, x) + R(t, x)$ be defined for $t \geq 0$, $\|x\| < a$, and suppose there exists a real-valued function $V(t, x)$ for which

$$A\|x\|^\alpha \leq V(t, x) \leq B\|x\|^\alpha,$$

$$V_t(t, x) + V_x(t, x) \cdot f(t, x) \leq -C\|x\|^{\alpha+m-1},$$

$$\|V_x(t, x)\| \leq L\|x\|^{\alpha-1},$$

where A, B, C, L are positive constants and $\alpha, m \geq 1$.

In the first paper the author shows that if $\alpha = 2$, $m = 1$, and $\|R(t, x)\| \leq \varphi(t)\|x\|$, where $\lim\sup_{t\to\infty} t^{-1} \int_0^t \varphi(s)ds$ is sufficiently small, then every solution satisfies for $t \geq 0$

$$\|x(t)\| \leq N\|x(0)\| \exp(-\nu t), \quad \nu > 0.$$

His proof is identical with the one by which the reviewer proved the same assertion under the assumption that $V(t, x)$ be a quadratic form in x and $\int^\infty \varphi(t)dt < \infty$ [J. London Math. Soc. **31** (1956), 208–212; MR **18**, 42].

The second paper is concerned with the case α, m general and $\|R(t, x)\| \leq \omega(t, \|x\|^m)$ where $\omega(t, y)$ is non-decreasing in y and $\omega(t, 0) \equiv 0$. Here the author shows that if the trivial solution of $y' = \omega(t, y)$ is asymptotically stable so is $x(t) \equiv 0$.

H. A. Antosiewicz (Los Angeles, Calif.)

3850:

Opial, Z. Sur une équation différentielle non linéaire du second ordre. Ann. Polon. Math. **8** (1960), 65–69.

Three theorems are proved concerning solutions of the two equations (1) $x'' + \varphi(x, x')x' + h(x) = e(t)$ and (2)

$x'' + (f(x) + g(x)x')x' + h(x) = e(t)$. In all three theorems it is assumed that φ, e, f, g and h are continuous, $xh(x) > 0$ for $x \neq 0$ and $\int_0^\infty |e(t)|dt < \infty$. Combinations of the following hypotheses are used in the theorems: (3) there exists a continuous function $m(x)$ defined for all x and such that $\varphi(x, u) - m(x)u > 0$ for all $x^2 + u^2 > 0$;

$$(4) \qquad 0 < \exp\left(\int_0^x m(u)du\right) \leq A < \infty;$$

$$(5) \qquad \lim_{|x|\to\infty} \int_0^x h(u) \exp\left(\int_0^u m(s)ds\right)du = \infty;$$

(6) $\varphi(x, u) > 0$ for all $x^2 + u^2 > 0$; (7) $\lim_{|x|\to\infty} \int_0^x h(u)du = \infty$; (8) $x \int_0^x f(u) \exp(\int_0^u m(s)ds)du > 0$ for $x \neq 0$. These hypotheses are called upon to prove (9) that if $x(t)$ is a solution, valid for all large t, of the equation being considered, then $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = 0$.

Theorem I: Conditions (3), (4) and (5) imply (9) for equation (1). Theorem II: Conditions (6) and (7) imply (9) for equation (1). Theorem III: Conditions (4), (5) and (8) imply (9) for equation (2). Similar hypotheses were used by Antosiewicz [J. London Math. Soc. **30** (1955), 64–67; MR **16**, 477] to show that solutions, and their derivatives, of (1) and (2), valid for all large t, are bounded for all large t. *W. R. Utz* (Columbia, Mo.)

3851:

Opial, Z. Sur les solutions de l'équation différentielle $x'' + h(x)x' + f(x) = e(t)$. Ann. Polon. Math. **8** (1960), 71–74.

In the real ordinary differential equation $x'' + h(x)x' + f(x) = e(t)$, let $g(x) = \int_0^x h(s)ds$. The author proves that if $f(x)$, $h(x)$ and $e(t)$ are continuous for all x and $t \geq 0$, if $|e(t)| \leq m$ for $t \geq 0$, if $\liminf_{x\to+\infty} f(x) > m$, $\limsup_{x\to-\infty} f(x) < -m$, and if for a positive number p sufficiently small

$$\liminf_{x\to+\infty} (g^2(x) - 2p|g(x)| - 4m|x|) > -\infty,$$

while $\lim_{x\to+\infty} g(x) = +\infty$ and $\lim_{x\to-\infty} g(x) = -\infty$, then all solutions of this differential equation defined for all large t are bounded. This equation has been treated earlier by S. Lefschetz [*Lectures on differential equations*, Princeton Univ. Press, Princeton, N.J., 1946; MR **8**, 68], K. Urabe [Math. Japon. **2** (1950), 23–26; MR **12**, 182] and the reviewer [Boll. Un. Mat. Ital. (3) **11** (1956), 28–30; MR **17**, 1208]. *W. R. Utz* (Columbia, Mo.)

3852:

Opial, Z. Sur la dépendance des solutions d'un système d'équations différentielles de leurs seconds membres. Application aux systèmes presque autonomes. Ann. Polon. Math. **8** (1960), 75–89.

Consider the system of real differential equations $X' = f(t, X)$, where the function f, defined on the product space of the reals and euclidean n-space, is continuous and sufficiently well-behaved to guarantee both the existence and uniqueness of solutions. If $X = X(t, P_0)$ is a solution of the system passing through the point $P_0 = (t_0, X_0)$, and J is the maximum interval of its validity as a solution, then for each compact interval $K \subset J$ with $t_0 \in K$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that each solution $Y(t, P_0)$ of the system $X' = f(t, X) + g(t)$, where $g(t)$ is a continuous vector function such that $\int_K |g(t)|dt \leq \delta$ and $X_0 = g(t_0)$, is valid in the interval K and satisfies the inequality $|X(t, P_0) - Y(t, P_0)| \leq \varepsilon$. This theorem is used to compare solutions of the autonomous system $X' = f(X)$ and the almost autonomous system $X' = f(X) + g(t, X)$. The latter system is called almost autonomous when $g(t, X)$ is such that for each continuous, bounded vector function $X(t)$ defined on (t_0, ∞) one has $\int_{t_0}^\infty |g(t, X(t))|dt < \infty$. By this comparison the author secures theorems concerning the limiting sets of solutions of the almost autonomous system. Similar theorems have been secured by L. Markus [*Contributions to the theory of nonlinear oscillations, Vol. 3*, pp. 17–29, Princeton Univ. Press, Princeton, N.J., 1956; MR **18**, 394] for asymptotically autonomous systems. *W. R. Utz* (Columbia, Mo.)

3853:

Tanrıkulu, Mahmut. The integral curves of the homogeneous differential equations in the neighbourhood of a singular point and a singular solution. Bull. Tech. Univ. Istanbul **12** (1960), 41–58. (Turkish summary)

The relative position of the regular solutions and singular solutions of the real differential equation $dr/d\varphi = \cot(f(\varphi) - \varphi)$ are studied in general and for several specific choices of $f(\varphi)$. The function $f(\varphi)$ is always continuous and satisfies the condition that for some k, $f(\varphi + 2\pi) = f(\varphi) + k\pi$. *W. R. Utz* (Columbia, Mo.)

3854:

Landis, E. M.; Petrovskiĭ, I. G. On the number of limit cycles of the equation $dy/dx = P(x, y)/Q(x, y)$, where P and Q are polynomials. Amer. Math. Soc. Transl. (2) **14** (1960), 181–199.

Translation of Mat. Sb. (N.S.) **43** **(85)** (1957), 149–168 [MR **19**, 746].

3855:

LaSalle, J. P. The extent of asymptotic stability. Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 363–365.

In studying the stability of a system, it is necessary, from the practical point of view, to have some idea of the region of asymptotic stability. The present paper gives some theorems related to this problem. Consider the system (1) $\dot{x} = X(x)$, where x, X are n-vectors, and where for each x^0 there exists a unique solution $x(t, x^0)$ of (1) with $x(0, x^0) = x^0$ and this solution depends continuously on x^0. Theorem 1: Let Ω be a bounded closed set such that $x^0 \in \Omega$ implies $x(t, x^0) \in \Omega$, $t \geq 0$. Suppose there exists a scalar function $V(x)$ such that $\dot{V}(x) = (\text{grad } V) \cdot X \leq 0$ in Ω. Let E be the set of points in Ω with $\dot{V}(x) = 0$ and let M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$. Two other theorems are given. One deals with the case when the set Ω can be constructed from $V(x)$ and the other gives conditions for the stability in the large of the set M above. The proofs are to appear elsewhere. *J. K. Hale* (Baltimore, Md.)

3856:

Bogusz, Władysław. Determination of stability regions of dynamic non-linear systems. Arch. Mech. Stos. **11** (1959), 691–713. (Polish and Russian summaries)

The author presents a detailed study of the solutions of the two-dimensional system $\dot{x}_1 = f(x_1, x_2)$, $\dot{x}_2 = g(x_1, x_2)$

under various assumptions concerning the Jacobian of f and g. Considerable use is made of Lyapunov functions.
R. E. Bellman (Santa Monica, Calif.)

3857:

Ronsmans, P. Étude des courbes caractéristiques d'un système d'équations différentielles non linéaires. Acad. Roy. Belg. Bull. Cl. Sci. (5) **45** (1959), 1031–1048.

The author studies the topological behavior of the characteristic curves for a differential system in the neighborhood of its singular points. The problem had already been solved for the case of simple singularities, and in this paper the method of proper values is generalized so as to obtain a topological analysis in the neighborhood of multiple singularities. The characteristic curves of the given system are solutions of an equation of the type $dy/dx=B(x, y)/A(x, y)$ and the singular points satisfy $A(x, y)=B(x, y)=0$. The problem of the topological behavior of these curves is solved in certain particular cases. Use is made of the slopes of the n lines given by $f_n(x, y)=0$ when f_n is the first non-vanishing term in the development of $f(x, y)=x\cdot B(x, y)-y\cdot A(x, y)$.
G. T. Whyburn (Charlottesville, Va.)

3858:

McKelvey, Robert. Solution about a singular point of a linear differential equation involving a large parameter. Trans. Amer. Math. Soc. **91** (1959), 410–424.

The differential equation

$$(*)\qquad \frac{d^2u}{dz^2}-[\lambda^2\phi^2(z)+\lambda z^{-1}\psi(z, \lambda)+z^{-2}\tau]u = 0$$

is studied in its dependence on a large complex parameter, λ, for various values of the complex variable z. It is assumed that z lies in a simply connected compact neighborhood R of the origin (the singular point), and that $\phi(z)\neq 0$ in R, $\psi(z, \lambda)$ being analytic for z in R, $|\lambda|>N$, while $\tau=$const.

The method of attack is to replace (*) by a related differential equation, identical with (*) up to and including terms of order λ^{-n}, $|\lambda|>N$; the related equation is constructed from the Whittaker equation, whose solutions and asymptotic expansions are well known. Finally, an integral equation connects the solutions of (*) with those of the related equation. Classical methods now yield the desired asymptotic expansions, complete with their regions of validity.
G. E. Latta (Stanford, Calif.)

3859:

Sabirova, H. On the solution of a linear differential equation of the type of Hill's equation by the small parameter method with accuracy to the second order inclusive. Izv. Akad. Nauk UzSSR. Ser. Fiz.-Mat. **1958**, no. 3, 67–77. (Russian. Uzbek summary)

The author derives the first three terms in the expansion $y(x)=\sum \mu^n y_n(x)$ of a solution of the equation $y''+(k^2+\mu f(x))y=0$ where f is holomorphic and of period 2π.
H. A. Antosiewicz (Los Angeles, Calif.)

3860:

Halanai, A. [Halanay, A.] An averaging method for systems of differential equations with lagging argument. Rev. Math. Pures Appl. **4** (1959), 467–483. (Russian)

Theorem 1: Consider the system

$$(1)\qquad \dot{x} = \varepsilon X[t, x(t), x(t-\tau)],$$

where x is an n-vector, $X(t, x, y)$ is bounded for $t\in[0, \infty)$, $x\in D$, $y\in D$, τ is a real constant,

$$\lim_{T\to\infty} T^{-1}\int_0^T X(t, x, y)dt = X_0(x, y)$$

for $x\in D$, $y\in D$, and for every $\eta>0$ there exists a $\delta(\eta)>0$ such that $|X(t, x_1, y_1)-X(t, x_2, y_2)|<\eta$, $|X_0(x_1, y_1)-X_0(x_2, y_2)|<\eta$, if $|x_1-x_2|<\delta$, $|y_1-y_2|<\delta$. Suppose also that the equations (2) $\dot{y}(t)=\varepsilon X_0[y(t), y(t)]$ have a unique solution satisfying $y(0)=x_0$. Then, if $x(t, \varepsilon)$, $y(t, \varepsilon)$ are solutions of (1), (2) respectively, with $x(0, \varepsilon)=y(0, \varepsilon)=x_0$, then for every T, η there is an $\varepsilon_0>0$ such that $|x(t, \varepsilon)-y(t, \varepsilon)|<\eta$ for all $t\in[0, T/\varepsilon]$, if $0<\varepsilon<\varepsilon_0$. This theorem is proved by using the ideas in Krasnosel'skiĭ and Kreĭn, Uspehi Mat. Nauk (N.S.) **10** (1955), no. 3 (65). 147–152 [MR **17**, 152] for the proof of a similar theorem for differential equations.

Theorem 2: Consider the system (3) $\dot{z}(t)=\varepsilon Z[t, z(t), z(t-\varepsilon\tau), \varepsilon]$, where $Z(t, u, v, \varepsilon)$ has continuous partial derivatives of the first order and $Z(t+T, u, v, \varepsilon)=Z(t, u, v, \varepsilon)$. Suppose $Z_0(u, v, \varepsilon)=T^{-1}\int_0^T Z(t, u, v, \varepsilon)dt$ and $Z_0(\zeta_0, \zeta_0, 0)=0$ for some ζ_0. Then, if the characteristic roots of the matrix $\partial Z_0(\zeta_0, \zeta_0, 0)/\partial u+\partial Z_0(\zeta_0, \zeta_0, 0)/\partial v$ have negative real parts, there exists an $\varepsilon_0>0$ such that, for $0<\varepsilon<\varepsilon_0$, system (3) has a periodic solution of period T which tends to ζ_0 as $\varepsilon\to 0$. This theorem is proved by using a transformation analogous to the one used by Bogolyubov and Mitropol'skiĭ [*Asimptotičeskie metody v teorii nelineĭnyh kolebaniĭ*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955; revised, 1958; MR **17**, 368; **20** #6812] for differential equations, and then using a Lyapunov function to show that a certain operator has a fixed point.
J. K. Hale (Baltimore, Md.)

3861:

Halanay, Aristide. Sur les systèmes d'équations différentielles linéaires à argument retardé. C. R. Acad. Sci. Paris **250** (1960), 797–798.

It is shown that a linear difference-differential equation with periodic coefficients has a periodic solution if it has a bounded solution.
E. Pinney (Berkeley, Calif.)

3862:

Šimanov, S. N. On the instability of the motion of systems with retardation. Prikl. Mat. Meh. **24** (1960), 55–63 (Russian); translated as J. Appl. Math. Mech. **24**, 70–81.

It is shown that known theorems of A. M. Lyapunov [*Obščaya zadača ob ustoičivosti dviženiya*, GITTL, Moscow-Leningrad, 1950; MR **12**, 612] and N. G. Četaev [*Ustoičivost' dviženiya*, GITTL, Moscow, 1955; MR **17**, 1087] concerning stability may be extended to systems with retardation:

$$dx_i(t)/dt = X_i(x_1(t+v), \cdots, x_n(t+v), t) \qquad (i = 1, \cdots, n),$$

where the $X_i(x_1(v), \cdots, x_n(v), t)$ are functionals defined for any piecewise continuous functions $x_i(v)$ defined on

the interval $-r \leq v \leq 0$, and $X_t(0, \cdots, 0, t) \equiv 0$. A first approximation criterion of instability for systems with retardation is obtained. *J. B. Diaz* (College Park, Md.)

3863:

Chin, Yuan-shun; Liu, Ying-ching; Wang, Lian. On the equivalence problem of differential equations and difference-differential equations in the theory of stability. Acta Math. Sinica **9** (1959), 333–363. (Chinese. English summary)

The authors consider the connection between the stability of the null solution of $dx/dt = (A+B)x(t)$ and that of $dx/dt = Ax(t) + Bx(t-\Delta(t))$. A number of sufficient conditions are given for the case where $\Delta(t)$ is sufficiently small. *R. E. Bellman* (Santa Monica, Calif.)

PARTIAL DIFFERENTIAL EQUATIONS

See also 3795, 3796, 3830.

3864:

Рихтмайер, Р. Д. [Richtmyer, Robert D.]. Разностные методы решения красвых задач. [Difference methods for initial-value problems.] Translated from the English by B. M. Budak and A. D. Gorbunov. Izdat. Inostr. Lit., Moscow, 1960. 262 pp. 10.80 r.

The English original [Interscience, New York, 1957] was reviewed as MR **20** #438.

3865:

Abian, Smbat; Brown, Arthur B. On the solution of an implicit first order partial differential equation. Rend. Circ. Mat. Palermo (2) 8 (1959), 271–296.

The partial differential equation considered has the form (1) $f(x, y_1, \cdots, y_{n-1}, z, \partial z/\partial y_1, \cdots, \partial z/\partial y_{n-1}, \partial z/\partial x) = 0$. Formulas are given, involving only quadratures and partial derivatives of f and of ω, for computing successive approximations to the solution of (1) passing through an initial manifold $x = x_0$, $z = \omega(y_1, \cdots, y_{n-1})$. The solution is obtained in parametric form: $y_p = \pi_p(x, t_1, \cdots, t_{n-1})$, $z = \pi_n(x, t_1, \cdots, t_{n-1})$. Several estimates of the error in the approximations are given.

L. M. Graves (Chicago, Ill.)

3866:

Boigelot, A.; Garnir, H. G. Nouvelles expressions des noyaux de Green relatifs aux opérateurs métaharmonique, des ondes et de la diffusion. Ricerche Mat. 7 (1958), 186–204.

Gli A. espongono un nuovo ed interessante metodo per introdurre i nuclei di Green per il problema di Dirichlet-Neumann (D.-N.) relativo agli operatori metaarmonico, delle onde e della diffusione. Esso si appoggia anzitutto su una formula di rappresentazione delle soluzioni deboli, nel senso delle distribuzioni, dell'equazione metaarmonica: (I) $-\Delta u + zu = f$, $f \in L^1_{loc}(\Omega)$, ottenuta facendo uso della soluzione fondamentale l_k dell'operatore Δ^k; precisamente, detto Ω un aperto di R^n e Γ la sua frontiera, si ha per ogni soluzione $u \in L^1_{loc}(\Omega)$ della (I), nell'aperto $\Omega_\varepsilon = \{x : d(x, \Gamma) > \varepsilon\}$, $\varepsilon > 0$: $u = u * \Delta^k[(1-\alpha_\varepsilon)l_k] + z^k u * [\alpha_\varepsilon l_k] - \sum_{j=0}^{k-1} z^j f * \Delta^{k-j-1}[\alpha_\varepsilon l_k]$ dove $\alpha_\varepsilon(|x|)$ è una funzione indefinitamente differenziabile, $=0$ per $|x| \geq \varepsilon$, $=1$ per $|x| \leq \varepsilon/2$.

Considerato allora il problema generalizzato di D.-N. relativo alla (I) (per l'impostazione del problema e la nomenclatura si vedano il libro di H. G. Garnir, *Les problèmes aux limites de la physique mathématique*, Birkhäuser, Basel-Stuttgart, 1958, e le note di H. G. Garnir e J. Gobert su Bull. Soc. Roy. Sci. Liège **26** (1957), 279–289; **27** (1958), 17–27, 119–127, e le relative recensioni [MR **21** #2816; **20** #1066; **21** #2817]), si ottiene per il relativo operatore di Green $G_z f$, l'espressione $G_z f = \int_\Omega K(x_0, x; z) f(x) dx$, con $K(x_0, x; z)$, nucleo di Green, funzione di x_0 e $x \in \Omega$, la quale per $d(x_0, \Gamma) > \varepsilon$ ha la forma

$$K(x_0, x; z) = G_z[\Delta^k(1-\alpha_\varepsilon)l_k(x_0 - w) + z^k \alpha_\varepsilon l_k(x_0 - w)] - \sum_{j=0}^{k-1} z^j \Delta^{k-j-1} \alpha_\varepsilon l_k(x_0 - x)$$

con k sufficientemente grande. Per l'operatore delle onde $-\Delta + a\partial_t^2 + b\partial_t + c$, a, b, c reali e $a > 0$, sempre per il problema di D.-N., l'operatore di Green ha l'espressione $\mathfrak{G}_\varphi f = \int_\Omega \mathfrak{H}(x_0, x; \varphi) f(x) dx$, il nucleo di Green $\mathfrak{H}(x_0, x; \varphi)$ essendo dato per $x_0 \in \Omega_\varepsilon$ e $x \in \Omega$ e k sufficientemente grande da

$$\mathfrak{H}(x_0, x; \varphi) = \mathfrak{G}_\varphi \Delta^k(1-\alpha_\varepsilon)l_k(x_0 - w) + \mathfrak{G}_{(a\partial_t^2 - b\partial_t + c)^k\varphi} \alpha_\varepsilon l_k(x_0 - w) - \sum_{j=0}^{k-1} (a\partial_t^2 - b\partial_t + c)^j \varphi(0) \Delta^{k-j-1} \alpha_\varepsilon l_k(x_0 - x).$$

Si ha poi un teorema analogo per $\mathfrak{G}_\varphi * m_t$. Infine per l'operatore della diffusione $-\Delta + b\partial_t + c$, b e c reali e $b > 0$, si ha $\mathfrak{C}_t f = \int_\Omega H(x_0, x; t) f(x) dx$ dove $H(x_0, x; t)$ è dato per $x_0 \in \Omega_\varepsilon$ e $x \in \Omega$ e $t > 0$ da

$$H(x_0, x; t) = \mathfrak{C}_t \Delta^k(1-\alpha_\varepsilon)l_k(x_0 - w) + (b\partial_t + c)^k \mathfrak{C}_t \alpha_\varepsilon l_k(x_0 - w)$$

per k sufficientemente grande; e un teorema analogo si ha per $\mathfrak{C}_t * m_t$. Gli A. riottengono anche da queste espressioni in modo rapido ed elegante le principali proprietà degli operatori e dei nuclei di Green. *E. Magenes* (Pavia)

3867:

Agudo, Fernando Roldão Dias; Wolf, František. Propriétés spectrales de l'opérateur

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} + d$$

à coefficients complexes. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. **25** (1958), 273–275.

Let A be the closure in $L^2(\Omega)$ of the Laplacian $\Delta = \sum \partial^2/\partial x_i^2$ acting on the space $\mathscr{D}(\Omega)$ of all scalar functions with compact supports which are infinitely differentiable in the whole space Ω. Then the perturbation

$$B = \sum a_i(x)\partial/\partial x_i + a(x),$$

with square integrable bounded coefficients defined in Ω, is A-compact, meaning that the image BS of every set $S = \{x; \|x\|^2 + \|Ax\|^2 \leq M^2\}$ is pre-compact. Also the whole positive real axis constitutes the continuous spectrum of $A+B$. In the rest of the plane the spectrum of $A+B$ is discrete and consists of characteristic numbers having finite multiplicities. *L. Nachbin* (Waltham, Mass.)

3868:

Virabyan, G. V. Spectral equivalence of two operators. Dokl. Akad. Nauk SSSR **128** (1959), 13–16. (Russian)

Let A [resp. B] be the selfadjoint extension of the operator $A(v_x, v_y, v_z)=(w_x, w_y, w_z)$, $w_x=v_x+\partial P_0/\partial y+i(\partial P_1/\partial x)$, $w_y=v_y-\partial P_0/\partial x+i(\partial P_1/\partial y)$, $w_z=i(\partial P_1/\partial z)$, $P_0=\partial v_y/\partial x-\partial v_x/\partial y$, $P_1=i(\partial v_x/\partial x+\partial v_y/\partial y)$, $P_0|_\Gamma=P_1|_\Gamma=0$ [resp. the operator $\Delta^{-1}(\partial^2/\partial z^2)$] defined suitably in the Hilbert space H_A [resp. H_B] of the vectors (v_x, v_y, v_z), with scalar product

$$((v_x, v_y, v_z), (w_x, w_y, w_z))_A = \iiint_\Omega (v_x w_x+v_y w_y+v_z w_z)dxdydz$$

[resp. functions u, with scalar product

$$(u, v)_B = \iiint_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\right)dxdydz],$$

where Ω is a bounded domain and Γ its boundary.

The author establishes certain properties of spectral equivalence between A and B. *C. Foiaş* (Bucharest)

3869:

Redheffer, Raymond. On entire solutions of nonlinear differential equations. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 1292–1294.

Theorems are stated having as conclusions that the entire solutions of $\Delta u \geq f(u, u^*)$, $u^*=|\mathrm{grad}\, u|$, are constant. For example, the following theorem is said to include, as special cases, many theorems in the literature of the equation $\Delta u=f(u)$. Let $f(u)$ be continuous and positive for all u and suppose that there exists a constant $\alpha>0$ such that $f(u)\geq(\alpha/u)\int_0^u f(t)dt$ when $u\gg 1$. Suppose $g(p)p^{n/(1-n)}$ is continuous, positive, nonincreasing for $p>0$ and is bounded away from 0 for p small. Suppose that there exists a constant $\alpha>0$ such that $p/g(p)\leq(\alpha/p)\int_0^p f(t)dt$ for $p\gg 1$. Let $F(y)=\int_0^y f(t)dt$, $G(p)=\int_1^p (p/g(p))dp$, and assume that $\int^\infty dy/G^{-1}(F(y))<\infty$. Then every entire solution of $\Delta u\geq f(u)g(u^*)$ is constant.

W. R. Utz (Columbia, Mo.)

3870:

Landis, E. M. Some questions in the qualitative theory of elliptic and parabolic equations. Uspehi Mat. Nauk **14** (1959), no. 1 (85), 21–85. (Russian)

Second order partial differential equations of elliptic and parabolic type are considered from the viewpoint of function theory. In each case the signs of the coefficients are assumed such that the maximum principle holds. Among the topics covered are growth properties, the Phragmén-Lindelöf theorem, the Harnack inequalities, and the inter-connection between growth of the solution and the number of regions on which it has constant sign. For elliptic equations, this last topic includes a generalization of the theorem of Hadamard on analytic functions. The present work, dealing only with two independent variables, is to be followed by a corresponding treatment of the case of n independent variables.

M. G. Arsove (Seattle, Wash.)

3871:

Protter, M. H. Unique continuation for elliptic equations. Trans. Amer. Math. Soc. **95** (1960), 81–91.

Simplifiant la méthode de Pederson [Comm. Pure Appl. Math. **11** (1958), 67–80; MR **20** #5350], l'A. établit des inégalités qui lui permettent de retrouver le théorème d'unicité pour l'équation elliptique générale du second ordre, et d'étendre ce résultat aux inéquations de la forme (1) $|\Delta^n u|\leq f(x, u, Du, \cdots, D^k u)$, avec $k\leq 3n/2$: si f est lipschitzienne et si $u\exp(2r^{-\beta})\to 0$ quand $r=\|x\|\to 0$ quel que soit $\beta>0$, alors toute solution de (1) est identiquement nulle dans un voisinage de l'origine. *J. Lelong* (Paris)

3872:

Schechter, Martin. On the Dirichlet problem for second order elliptic equations with coefficients singular at the boundary. Comm. Pure Appl. Math. **13** (1960), 321–328.

In the domain $\mathscr{D}$ a linear second-order elliptic equation with n independent variables is considered, whose coefficients are unbounded as a point approaches a part Γ of the boundary of $\mathscr{D}$ and are Hölder continuous in any closed subset of $\overline{\mathscr{D}}$ not containing Γ. The paper gives sufficient conditions on the behavior of the coefficients of the equation in the neighborhood of Γ, in order that the Dirichlet problem in $\mathscr{D}$ be soluble for arbitrary continuous boundary function. The proof relies on bounds and a priori estimates of Schauder.

O. A. Oleĭnik (Moscow)

3873:

Volkov, D. M. On the uniqueness of the classical energy integral. Vestnik Leningrad. Univ. **1953**, no. 5, 3–13. (Russian)

Consider, in m-space $x_1, \cdots, x_m$, functions $u(x_1, \cdots, x_m, t)$ and $v(x_1, \cdots, x_m, t)$ satisfying the wave equation. Form the expression

$$J_1 = \int_D \{g^{ik}u_i v_k+g^{ti}u_t v_i+g^{it}u_i v_t+g^{tt}u_t v_t+g^{0t}uv_t + g^{t0}u_t v+g^{0i}u_i v+g^{i0}uv_i+g^{00}uv\}dx+\int_S Huvds,$$

where the first integral is taken over some region D of space, the second over its boundary S. Here $u_i=\partial u/\partial x_i$, $u_t=\partial u/\partial t$, and similarly for v; the coefficients $g^{ik}, g^{it}, \cdots$ are some functions of $x_1, \cdots, x_m$.

On the boundary S the functions u, v are subjected to one of the following 4 conditions: (1) $u=0, v=0$; (2) $\partial u/\partial n=0$, $\partial v/\partial n=0$; (3) $\partial u/\partial n+hu=0$, $\partial v/\partial n+hv=0$; (4) $\partial u/\partial n+hu_{tt}=0$, $\partial v/\partial n+hv_{tt}=0$.

In each case the question is completely answered, for which $g^{ik}, g^{it}, \cdots$ the expression J_1 remains invariant for varying t (the u, v entering in J_1 being arbitrary functions satisfying the above-stated requirement).

P. K. Raševskiĭ (RŽMat **1954** #1657)

3874:

Barancev, R. G. Boundary value problems for the hyperbolic equation $u_{xx}-K(x)u_{tt}=0$ in the strip $0\leq x\leq 1$ with degeneration or singularity on the boundary. I. Expansion theorems. Vestnik Leningrad. Univ. **14** (1959), no. 19, 13–35. (Russian. English summary)

Although problems closely related to the one of this paper have been previously considered by the author [Dokl. Akad. Nauk SSSR **114** (1957), 919–922; same Vestnik **13** (1958), no. 19, 19–38; MR **19**, 865; **21** #200],

this one bears restating: to solve the equation $u_{xx}-K(x)u_{tt}=0$ under the conditions $x\in[0,1]$, $u|_{t=l(x)}=p(x)$, $u_t|_{t=l(x)}=q(x)$ if $|l'(x)|<[K(x)]^{1/2}$, $u_t|_{t=l(x)}$ not given if $l'(x)\equiv[K(x)]^{1/2}$; $u(0,t)\cos\xi+u_x(0,t)\sin\xi=0$, $0<\xi\leqq\pi$; $u(1,t)\cos\eta+u_x(1,t)\sin\eta=0$, $0<\eta\leqq\pi$. The orientation $l(x)$ is defined in MR **21** #200. In this paper $K(x)=x^\alpha(1-x)^\beta\bar{K}(x)$, where $\bar{K}(x)$ is a positive twice differentiable function in $[0,1]$ and $2>\alpha>-1$ for $\xi\neq\pi$, $2>\beta>-1$ for $\eta\neq\pi$, $2>\alpha>-2$ for $\xi=\pi$, $2>\beta>-2$ for $\eta=\pi$. With the solution $u(x,t)$ expressed as an expansion in terms of the eigenvalues λ_n and eigenfunctions $B_n(x)$ of a Sturm-Liouville problem for the differential equation

$$(*)\qquad B_n''+\lambda_n^2KB_n=0,$$

the functions $p(x)$ and $q(x)$ given on the curve $t=l(x)$ also possess series expansions of this type. Convergence of these series is established under conditions appropriate for the respective cases $|l'(x)|<[K(x)]^{1/2}$, $l'(x)\equiv[K(x)]^{1/2}$. Proofs of the theorems are based on asymptotic formulas for the solution of (*) obtained by the method of A. A. Dorodnicyn [Uspehi Mat. Nauk **7** (1952), no. 6 (52), 3–96; MR **14**, 876]. *R. N. Goss* (San Diego, Calif.)

3875:

Barancev, R. G. Boundary value problems for the hyperbolic equation $u_{xx}-K(x)u_{tt}=0$ in the strip $0\leqq x\leqq 1$ with degeneration or singularity on the boundary. II. Study of solutions. Vestnik Leningrad. Univ. **15** (1960), no. 1, 14–33. (Russian. English summary)

In this part conditions are determined in order that

$$u(x,t)=2\operatorname{Re}\{\sum_{n=1}^{\infty}c_nB_n(x)\exp(-i\lambda_nt)\}$$

may be a solution in the classical or in some generalized sense of the boundary-value problem studied in part I [cf. preceding review]. The series is transformed into a slightly different form, and the asymptotic behavior of the coefficients is studied under various assumptions about the smoothness of $K(x)$, $l(x)$, $p(x)$, $q(x)$ and the consistency of the boundary conditions at the points where the curve $t=l(x)$ intersects the lines $x=0$ and $x=1$. The main results are theorems setting forth the conditions for uniform convergence of the series for u and for its derivatives in the characteristic directions (i.e., along l) for the cases $l'(x)\equiv[K(x)]^{1/2}$ and $|l'(x)|<[K(x)]^{1/2}$. These series are studied both in the strip $\{0<x_1\leqq x\leqq x_2<1\}$ and in the strip $\{0\leqq x\leqq 1\}$. *R. N. Goss* (San Diego, Calif.)

3876:

Lees, Milton. The Goursat problem. J. Soc. Indust. Appl. Math. **8** (1960), 518–530.

In this paper the author proves some facts concerning existence, uniqueness and finite-difference approximation for a quasi-linear Goursat problem

$$\frac{\partial^2 z}{\partial x\partial y}=A_1(x,y,z)p+A_2(x,y,z)q+A_3(x,y,z).$$

Its initial data are prescribed on two intersecting characteristics. For the existence, he assumes that the functions A_i $(i=1,2,3)$ are measurable in (x,y) for each z, bounded for bounded z, and continuous in z for each (x,y). First he gives an existence theorem for linear equations with bounded measurable coefficients. The solution is obtained as a limit of a sequence of finite-valued step functions, where the height of the steps for each member of the sequence is determined by an algorithm suitable for practical computation.—The result for linear equations and the fixed-point theorem of Schauder can be employed to prove an existence theorem for the quasi-linear equation. The solution is unique when the A_i satisfy a Lipschitz condition with respect to z. The author recovers the main results of R. T. Dames' paper [J. Math. and Phys. **38** (1959/60), 42–67; MR **22** #2016] for one of the most important difference schemes by using a quite different approach. He gets somewhat sharper results under weaker hypotheses. *M. Pinl* (Cologne)

3877:

Carroll, Robert. Quelques problèmes de Cauchy singuliers. C. R. Acad. Sci. Paris **251** (1960), 498–500.

The note is devoted to the existence and uniqueness of solution for the singular Cauchy problem for some equations generalizing the Euler-Poisson-Darboux equation. In the notation of L. Schwartz, consider the equation

$$\frac{\partial^2}{\partial t^2}\omega_x^k(t)+\left[\frac{k}{t}+\alpha(t)\right]\frac{\partial}{\partial t}\omega_x^k(t)+[\mathscr{A}_x+\gamma(t)\delta]*\omega_x^k(t)=0$$

and the initial conditions

$$\omega_x^k(\tau)=T\in\mathscr{S}',\quad \frac{\partial}{\partial t}\omega_x^k(\tau)=0,$$

where $k\geqq 0$, $t\to\omega_x^k(t)\in\mathscr{E}_t^2(\mathscr{S}_x')$, $0\leqq\tau\leqq t\leqq b<\infty$, $\alpha\in C'[0,b]$ and $\gamma\in C^0[0,b]$. Let $A(y)=\mathscr{F}\mathscr{A}_x$ be the Fourier transform of the differential operator $\mathscr{A}_x$ and assume that $A(y)$ is real and there are $a>0$ and $R_0\geqq 0$ such that $A(y)\geqq a$ for $|y|>R_0$. By a spatial Fourier transform, it is shown that this problem is (uniformly) well posed and that its solution is naturally expressed by a convolution with T. *L. Nachbin* (Waltham, Mass.)

3878:

Friedman, Avner. Parabolic equations of the second order. Trans. Amer. Math. Soc. **93** (1959), 509–530.

The paper deals with a special form of parabolic equation

$$Lu\equiv\sum_{i=1}^{n}a_i(x,t)\frac{\partial^2u}{\partial x_i^2}+\sum_{i=1}^{n}b_i(x,t)\frac{\partial u}{\partial x_i}+c(x,t)u-\frac{\partial u}{\partial t}=f(x,t).$$

For rectangular regions the Green's function is constructed and its properties studied. Using the Green's function a Harnack inequality is established, and theorems analogous to the familiar Harnack theorems for harmonic functions. These results are applied to the construction of the solution of the first boundary problem by the method of balayage, under very general assumptions about the boundary of the region. The first boundary problem is solved also for the non-linear equation $L(u)=f(x,t,u)$, under certain restrictions on the order of growth of f with respect to u. *O. A. Oleĭnik* (Moscow)

3879:

Šabat, B. V. Geometric interpretation of the concept of ellipticity. Uspehi Mat. Nauk **12** (1957), no. 6 (78), 181–188. (Russian)

Clarification of the geometric meaning of the concept

of ellipticity for linear systems of the form $v_y = au_x + bu_y$, $-v_x = du_x + cu_y$, as well as certain nonlinear systems. It is also shown that from ellipticity in the sense of Lavrent'ev of nonlinear systems of the form $F_i(u_x, u_y, v_x, v_y)$, $i = 1, 2$, under the condition $\partial(F_1, F_2)/\partial(v_x, v_y) \neq 0$, follows ellipticity in the sense of Petrovskiĭ; supplementary conditions are given sufficient for the converse implication to hold.
B. V. Boyarskiĭ (RŽMat **1959** #4738)

3880:

Antohi-Teodorescu, Veronica. Solution élémentaire de l'équation aux dérivées partielles linéaires d'ordre six, du type hyperbolique, à caractéristique triple. Acad. R. P. Romîne. Stud. Cerc. Mat. **11** (1960), 249–261. (Romanian. Russian and French summaries)

Earlier investigations [An. Univ. "C.I. Parhon" Bucureşti. Ser. Şti. Nat. **6** (1957), no. 15, 9–24, **7** (1958), no. 17, 9–21; MR **20** #3377, 4707] are extended from a partial differential equation of order four to one of order six.
A. Erdélyi (Pasadena, Calif.)

3881:

Yamaguti, Masaya; Kasahara, Koji. Sur le système hyperbolique à coefficients constants. Proc. Japan Acad. **35** (1959), 547–550.

The authors consider the first order constant coefficient system $\partial u/\partial t - \sum A_j \partial u/\partial x_j - Bu = 0$, and show that certain conditions known to be sufficient for the system to be strongly hyperbolic are also necessary for this to be true.
R. W. McKelvey (Los Angeles, Calif.)

3882:

Miles, E. P., Jr. The analytic Cauchy problem for the iterated wave equation. Portugal. Math. **18** (1959), 111–119.

L'auteur construit un nouveau système de base des solutions polynomiales des équations de Laplace et des ondes itérée. Ce système conduit à une solution explicite simple du problème de Cauchy pour l'équation des ondes itérée $\Box^m u = 0$, $\Box = -D_t^2 + \sum_{i=1}^n D_{x_i}^2$, avec des données $D_t^i u(x, 0) = f_i(x)$ $(i = 1, \cdots, 2m-1)$ polynomiales ou polyharmoniques.
H. G. Garnir (Liège)

3883:

Fourès-Bruhat, Yvonne. Propagateurs et solutions d'équations homogènes hyperboliques. C. R. Acad. Sci. Paris **251** (1960), 29–31.

Let L be a linear hyperbolic operator which admits elementary solutions: $L_{x'} E_\pm(x, x') = \delta(x' - x)$, and E_+, E_- are distributions with support compact towards the past and future, respectively. Then every distribution u such that $Lu = 0$ is a convolution of $E_+ - E_-$ with a distribution of compact support.
P. Ungar (New York)

3884:

Šilov, G. E. On a theorem of Phragmén-Lindelöf type for a system of partial differential equations. Amer. Math. Soc. Transl. (2) **14** (1960), 201–216.

Translation of Trudy Moskov. Mat. Obšč. **5** (1956), 353–366 [MR **18**, 657].

3885:

Pliś, A. On the uniqueness of the non-negative solution of the homogeneous mixed problem for a system of partial differential equations. Ann. Polon. Math. **7** (1960), 255–258.

La solution $\mathbf{u}(x, t)$ du système

$$D_t \mathbf{u} = \sum_{\alpha,\beta} A_{\alpha\beta}(x, t) D_{\alpha\beta}^2 \mathbf{u} + \sum_\gamma A_\gamma(x, t) D_\gamma \mathbf{u} + B(x, t)\mathbf{u},$$

où les $A_{\alpha\beta}$, A_γ et B sont des matrices relativement régulières, est nulle dans $\{(x, t) : |x_1| \leq 1, \cdots, |x_n| \leq 1, t \in]0, 1[\}$, si elle s'annule sur $t = 0$ et lorsque $\max_{k=1,\ldots,n} |x_k| = 1$, $t \in]0, 1[$, pour autant que ses composantes soient positives. On peut même laisser cette restriction de côté dans le cas particulier de l'équation scalaire de la chaleur.
H. G. Garnir (Liège)

3886:

Milicer-Grużewska, H. Le théorème d'unicité de la solution du système parabolique des équations linéaires avec les coefficients höldériens. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 593–599. (Russian summary, unbound insert)

The author demonstrates the uniqueness of the solution to the Cauchy problem for a system of linear parabolic partial differential equations with coefficients satisfying certain Hölder conditions in $E^n \times [0, T]$, under the assumption that the solutions belong to the class Z_1 of functions satisfying: (1) the first time derivative is continuous on $E^n \times (0, T)$; (2) the spatial derivatives of order less than or equal to the order of the system are continuous and Lebesgue integrable on $E^n \times (0, T)$.
J. Elliott (New York)

3887:

Shirota, Taira. A remark on the abstract analyticity in time for solutions of a parabolic equation. Proc. Japan Acad. **35** (1959), 367–369.

Consider the equation (1) $D_t u = A(t)u$, $t \in (-1, 1)$, where, for each t, $A(t)$ is a linear differential operator, elliptic on E^m. The coefficients, as functions of t and x, are assumed uniformly analytic in t for $x \in E^m$. The author proves that the solution u of (1) is an analytic function on a neighborhood of 0 into $L_2(E^m)$ whenever u is a continuously differentiable function on $(-1, 1)$ into $L_2(E^m)$.
J. Elliott (New York)

3888:

Aronson, D. G. On the initial value problem for parabolic systems of differential equations. Bull. Amer. Math. Soc. **65** (1959), 310–318.

The paper studies the Cauchy problem for a system parabolic in the sense of I. G. Petrovskiĭ [Byull. Moskov. Gos. Univ. (A) **1** (1938), no. 7, 1–74], with the assumption that the coefficients of the system are uniformly continuous functions of time and satisfy a uniform Hölder condition in the space coordinates. The fundamental solution of the system is constructed. It is shown that the Cauchy problem has a unique solution in the class of functions which, together with their derivatives to order m inclusive (m is the parabolic weight of the system), satisfy a Hölder condition, and are subjected to certain natural restrictions on the order of growth at infinity. This solution of the Cauchy problem is represented in terms of the fundamental solution, in case the initial

function and the non-homogeneous term satisfy certain continuity conditions and have the required behavior at infinity. *O. A. Oleĭnik* (Moscow)

3889:
Èĭdel'man, S. D. On a class of regular systems of partial differential equations. Uspehi Mat. Nauk **13** (1958), no. 4 (82), 205–209. (Russian)

The paper deals with systems of the form

$$\frac{\partial}{\partial t}\left[\frac{\partial u}{\partial t}-P\left(t,\frac{1}{i}\frac{\partial}{\partial x}\right)u\right] = R\left(t,\frac{1}{i}\frac{\partial}{\partial x}\right)u,$$

where $\partial u/\partial t-P(t,(1/i)\partial/\partial x)u=0$ is a system parabolic in the sense of I. G. Petrovskiĭ. The maximal order of differentiation in P is $2b$; R is an arbitrary linear differential operator of order not higher than $2b$. It is proved that systems of this form are regular [the definition of regularity is given in I. M. Gel'fand and G. E. Šilov, same Uspehi **8** (1953), no. 6 (58), 3–54; Amer. Math. Soc. Transl. (2) **5** (1957), 221–274; MR **15**, 867; **18**, 736]. The Cauchy problem for regular systems was studied earlier [A. G. Kostyučenko and G. E. Šilov, ibid. **9** (1954), no. 3 (61), 141–148; Amer. Math. Soc. Transl. (2) **5** (1957), 275–283; MR **16**, 253; **18**, 743]. *O. A. Oleĭnik* (Moscow)

3890:
Slobodeckiĭ, L. N. Remarks on "Uniqueness of solution for Cauchy's problem for quasi-linear symmetric systems". Uspehi Mat. Nauk **15** (1960), no. 1 (91), 262. (Russian)

In an article by the present author and M. I. Hramova [Uspehi Mat. Nauk **11** (1956), no. 4 (70), 155–162; MR **19**, 864] the hypothesis was made, among others, that the function $F(z)$ considered there is non-decreasing. The present article shows that this hypothesis is unnecessary.

3891:
Wloka, Josef. Über die Anwendung der Operatorenrechnung auf partielle Differential-Differenzengleichungen mit mehreren Differenzen. Arch. Math. **11** (1960), 23–28.

The main result is a uniqueness theorem for the solution to a partial difference-differential equation.
E. Pinney (Berkeley, Calif.)

POTENTIAL THEORY

See also 3692a–d, 3765, 3878.

3892:
Schiffer, M. Fredholm eigen values of multiply-connected domains. Pacific J. Math. **9** (1959), 211–269.

In einer früheren Untersuchung [imselben J. **7** (1957), 1187–1225; MR **21** #3684] hat Verf. das Fredholmsche Eigenwertproblem für den potentialtheoretischen Kern im Falle eines einfachzusammenhängenden ebenen Gebietes untersucht und Anwendungen gemacht auf verschiedene Fragen der konformen Abbildung und der Potentialtheorie. Hier wird diese Untersuchung auf mehrfachzusammenhängende Gebiete übertragen. $\tilde{D}$ bezeichnet ein Gebiet der abgeschlossenen Ebene, das den unendlichfernen Punkt enthält und von N dreimal stetig differenzierbaren Kurven C_j berandet wird. Das Innere von C_j wird mit D_j und die Vereinigung der D_j mit D bezeichnet. Der Kern ist $k(z,\zeta)=-\partial\log|\zeta-z|/\partial n$, $\zeta\in C=\bigcup C_j$. Zur Diskussion steht das Eigenwertproblem

$$\varphi_\nu(z)=\frac{\lambda_\nu}{\pi}\int_C k(z,\zeta)\varphi_\nu(\zeta)|d\zeta|,\quad z\in C.$$

Die rechte Seite definiert für $z\in D$ "eine" harmonische Funktion h_ν in D, für $z\in\tilde{D}$ eine harmonische Funktion in $\tilde{D}$, $\nu=1,2,\cdots$. Diesen sind die analytischen Funktionen $v_\nu=\partial h_\nu/\partial z$ in D bzw. $\tilde{v}_\nu=\partial\tilde{h}_\nu/\partial z$ in $\tilde{D}$ zugeordnet, die auf C noch stetig und Eigenfunktionen in bezug auf die Hilbertsche Integraltransformation sind. Sie bilden in bezug auf das Skalarprodukt $\int_D f\bar{g}\,dxdy$ bzw. $\int_{\tilde{D}}$ ein Orthogonalsystem, welches vollständig ist für die in D bzw. $\tilde{D}$ quadratisch integrierbaren holomorphen Funktionen mit eindeutiger Integralfunktion.

Die dielektrische Greensche Funktion $g_\varepsilon(z,\zeta)$ wird definiert als das elektrostatische Potential einer Punktquelle in ζ, wenn die Gebiete D_j durch isotrope Medien mit der Dielektrizitätskonstanten ε ausgefüllt sind. Das dieser elektrostatischen Situation entsprechende Randwertproblem hat genau eine Lösung, die sich folgendermassen durch die h_ν und $\tilde{h}_\nu$ darstellen lässt:

$$g_\varepsilon(z,\zeta)=-\log|\zeta-z|+2\pi(1-\varepsilon)\sum_1^\infty\frac{l_\nu(z)l_\nu(\zeta)}{(1+\rho_\nu)(1+\varepsilon\rho_\nu)},$$

wo $\rho=(\lambda+1)/(\lambda-1)$ und $l_\nu=h_\nu$ in D bzw. $l_\nu=\tilde{h}_\nu$ in $\tilde{D}$ ist. Diese Darstellung verlangt die Kenntnis der Eigenfunktionen h_ν und $\tilde{h}_\nu$; für die explizite Bestimmung von g_ε ist es zweckmässiger, die folgenden "geometrischen" Funktionale von D zu betrachten:

$$\Gamma^{(n)}(z,\zeta)=-(2\pi)^{-1}\int_D(\nabla_\eta\Gamma^{(n-1)}(\eta,z)\cdot\nabla_\eta\log|\eta-\zeta|)d\tau_\eta,$$

$$\Gamma^{(0)}(z,\zeta)=-\log|\zeta-z|\quad(n=1,2,\cdots).$$

Wird

$$M_k=\sum_1^\infty(-1)^k\binom{k}{\sigma}2^{k-\sigma+1}\Gamma^{(k-\sigma+1)}$$

gesetzt, so gilt

$$g_\varepsilon(z,\zeta)=-\log|z-\zeta|+\sum_{k=1}^\infty\left(\frac{\varepsilon-1}{\varepsilon+1}\right)^{k+1}M_k(z,\zeta).$$

Die dielektrische Greensche Funktion g_ε hängt mit der konformen Abbildung des Gebietes $\tilde{D}$ zusammen. Es definiere $\tilde{\varphi}(z,\zeta)$ bei festem $\zeta\in\tilde{D}$ die konforme Abbildung von $\tilde{D}$ auf die längs konzentrischen Kreisbogen um 0 aufgeschlitzte Ebene, sodass ζ in ∞ und ∞ in 0 übergeht mit $\lim_{z\to\infty}z\tilde{\varphi}(z,\zeta)=1$. Dann gilt $\lim_{\varepsilon\to 0}g_\varepsilon(z,\zeta)=\log|\tilde{\varphi}(z,\zeta)|$. Definiert $\tilde{\psi}(z,\zeta)$ in analoger Weise die konforme Abbildung von $\tilde{D}$ auf die radial geschlitzte Ebene, so ist $\lim_{\varepsilon\to\infty}g_\varepsilon(z,\zeta)=\log|\tilde{\psi}(z,\zeta)|$. Schliesslich gibt es, bei festem $\zeta\in\tilde{D}$, eine schlichte Funktion $\tilde{f}_\varepsilon$ in $\tilde{D}$, sodass $\log|\tilde{f}_\varepsilon(z,\zeta)|=g_\varepsilon(z,\zeta)$ ist. g_ε definiert also eine einparametrige Schar konformer Abbildungen von $\tilde{D}$, die zwischen den kanonischen Schlitzabbildungen interpoliert, die aber in bezug auf konforme Abbildungen von $\tilde{D}$ nicht invariant mit g_ε verknüpft ist.

$\tilde{\Sigma}$ sei die Klasse der in $\tilde{D}$ harmonischen Funktionen mit endlichem Dirichlet-Integral, die in ∞ verschwinden und eine eindeutige Integralfunktion haben, Σ die Klasse der in D harmonischen Funktionen (mit endlichem Dirichlet-Integral), welche gewisse N lineare Randbedingungen

erfüllen. $\tilde{D}(\tilde{h}, \tilde{H})$ bzw. $D(h, H)$ bezeichnet die Dirichletsche Bilinearform auf $\tilde{\Sigma}$ bzw. Σ. Die Bilinearform $\pi_\varepsilon(h, H)$ auf Σ ist durch den Ausdruck

$$\frac{1}{\pi}\int_C\int_C g_\varepsilon(z, \zeta)\frac{\partial h(z)}{\partial n}\cdot\frac{\partial H(\zeta)}{\partial n}|dz||d\zeta|$$

definiert, ganz entsprechend die Form $\tilde{\pi}_\varepsilon(\tilde{h}, \tilde{H})$ auf $\tilde{\Sigma}$. Ist ϕ die Abbildung von Σ in $\tilde{\Sigma}$, welche die Normalableitungen längs C invariant lässt, so ist $\tilde{\pi}_\varepsilon(\phi h, \phi H) = \pi_\varepsilon(h, H)$; lässt ψ die Randwerte auf C fest, so ist

$$\varepsilon\tilde{D}(\psi h, \psi H) - \varepsilon\tilde{\pi}_\varepsilon(\psi h, \psi H) = D(h, H) - \varepsilon^{-1}\pi_\varepsilon(h, H).$$

Ferner gilt

$$\pi_\varepsilon(h, H) = \varepsilon(\tilde{\pi}_\varepsilon(\phi h, \psi H) - \tilde{D}(\phi h, \psi H)),$$
$$D(h, H) = -\tilde{D}(\phi h, \psi H).$$

Die Bedeutung dieser Resultate liegt darin, dass sie gestatten eine Randwertaufgabe in $\tilde{D}$ auf eine solche in D zurückzuführen [Schiffer und Szegö, Trans. Amer. Math. Soc. **67** (1949), 130–205; MR **11**, 515]. So folgt z.B. für jedes $h \in \Sigma$ die Ungleichung

$$\tilde{D}(\psi h, \psi h) \geq \left(\frac{D(h, h)}{\pi_\varepsilon(h, h)} - \frac{1}{\varepsilon}\right) D(h, h),$$

wo das Gleichheitszeichen genau dann gilt, wenn $h = h_\nu$, $\nu = 1, 2, \cdots$ ist. Entsprechendes gilt für ϕ an Stelle von ψ und für die inversen Abbildungen. Besonders zweckmässig werden die Ungleichungen für $\varepsilon = 1$, wo $g_\varepsilon = -\log|\zeta - z|$ wird und die Bilinearformen leicht berechnet werden können. Man kann die Formen D und π_ε auch dazu benützen, um die Fredholmschen Eigenwerte abzuschätzen. Daraus ergibt sich, dass beim Uebergang von $C = \{C_j\}_{j=1}^N$ zu einem "kleinern" Kurvensystem $C^* = \{C_j\}_{j=1}^{N^*}$, $N^* < N$, sich der kleinste positive und nichttriviale Eigenwert λ_1 nie vergrössern kann.

Nun werden Variationsformeln für die Eigenwerte λ_ν und die Fredholmsche Determinante $\Delta(\lambda)$ des gegebenen Eigenwertproblems aufgestellt, wenn das Kurvensystem C durch Abbildungen der Form $z^* = z + \alpha/(z - z_0)$, $z_0 \notin C$ variiert wird. Es ergibt sich, dass $\Delta(\lambda)$ oberhalbstetig ist, wenn das Kurvensystem C im Sinne von Caratheodory konvergiert. Die Arbeit schliesst mit der Untersuchung eines Extremalproblems für $\Delta(1)$ im Zusammenhang mit der konformen Abbildung auf Kreisbereiche.

A. Pfluger (Zürich)

3893:

Poritsky, H. Potential of a charged cylinder between two parallel grounded planes. J. Math. and Phys. **39** (1960/61), 35–48.

The potential sought here is the function $V(x, y)$ harmonic in the domain $|x| < \pi/2$, $x^2 + y^2 > a^2$, where $0 < a < \pi/2$, such that $V = 0$ on the lines $x = \pm\pi/2$ and $V = 1$ on the circle $x^2 + y^2 = a^2$. Different types of series representations of an analytic function whose real part is $V(x\ y)$ are employed to arrive at an infinite set of linear algebraic equations in infinitely many unknowns. Those unknowns are the coefficients in a series from which the function V can be obtained. Questions of convergence are treated.

R. V. Churchill (Ann Arbor, Mich.)

3894:

Diaz, J. B. Upper and lower bounds for quadratic integrals, and at a point, for solutions of linear boundary value problems. Boundary problems in differential equations, pp. 47–83. Univ. of Wisconsin Press, Madison, 1960.

This is an expository article "intended as a quick introduction to several methods for obtaining reliable, precise, numerically computable upper and lower bounds for a large class of problems". Starting with the proof of the Schwarz inequality, which is the basis of the whole theory, an elementary exposition of the methods is given in connection with the estimation of the Dirichlet integral, and the value at a given point, of a harmonic function defined as the solution of a Dirichlet problem. (In particular this involves the estimation of the capacity C of a conductor.) Rather special but important formulas for the bounds of C of a star shaped surface are also introduced, concluding with the newest estimate of C for a cube.

T. Kato (Tokyo)

3895:

Nitsche, Johannes C. C. On the constant of E. Heinz. Rend. Circ. Mat. Palermo (2) **8** (1959), 178–181.

Let $x(\xi, \eta)$ and $y(\xi, \eta)$ be two harmonic functions in the unit disc, and suppose that the pair (x, y) gives a one-to-one mapping of the unit disc onto itself. Let μ be the value at the origin of $(x_\xi^2 + x_\eta^2 + y_\xi^2 + y_\eta^2)$. Then E. Heinz has shown that $\mu > 0.358$. In the present note the author shows that $\mu \geq 0.64$. The best estimate for μ is not yet known.

H. L. Royden (Stanford Calif.)

3896:

Kalik, Carol. La solution d'un problème aux limites pour l'équation biharmonique. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. **9** (1958), 135–148. (Romanian. Russian and French summaries)

L'A., usandovi il fruttuoso metodo di M. Picone concernente l'impiego dei sistemi integrali vettoriali di Fischer-Riesz [vedasi, ad. es., Convegno Mat. Appl., Rome, 1936, pp. 1–36; Zanichelli, Bologne, 1939], ne da la seguente applicazione di una serie di teoremi di completezza connessi all'integrazione dell'equazione $\Delta_4 u = f$ [G. Fichera, Giorn. Mat. Battaglini (4) **77** (1947), 184–199; MR **10**, 298]. La successione completa di vettori $\{v_i, -\Delta v_i + (1-\sigma)\rho_0^{-1}\partial v_i/\partial \nu\}$ (nel senso di Hilbert) su una curva Γ di raggio di curvatura positivo ρ_0 e che ne è la frontiera di un dominio limitato Ω, ove si è posto $\{v_n\} = \{\alpha_n\}V\{\beta_n\}$; $\alpha_1 = -\rho\cos\varphi$, $= -\rho\sin\varphi$; $\alpha_n = (n(n-1))^{-1}\rho^n \cos n\varphi$, $= (n(n-1))^{-1}\rho^n \sin n\varphi$ $(n = 2, 3, \cdots)$; $\beta_0 = \rho^2$; $\beta_n = -(n(n+1))^{-1}\rho^{n+2}\cos n\varphi$, $= -(n(n+1))^{-1}\rho^{n+2}\sin n\varphi$ $(n = 1, 2, \cdots)$; essendovi $V = r\ln r$, ρ e φ—le coordinate polari col centro in un punto interiore al dominio Ω, σ—la costante di Poisson e ν—la normale interiore rispetto a Γ, ne costituisce la base per la soluzione del problema

$$\Delta^2 u = \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2\partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0 \quad \text{su } \Omega,$$

$$u = f_1, \quad -\Delta u + \frac{1-\sigma}{\rho_0}\frac{\partial u}{\partial \nu} = f_2 \quad \text{su } \Gamma,$$

nell'ipotesi che f_1 e f_2 sono due funzioni dati su Γ, di quadrato sommabile.

D. Mangeron (Iaşi)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See also 3742, 3743, 3744, 3745, 3768.

3897:

Myrberg, P. J. The solution of the functional equation

$$\varphi(u+v)+\varphi(u-v) = R(\varphi(u), \varphi(v))$$

in which R is rational. Arkhimedes **1959**, no. 1, 13–16. (Finnish. German summary)

All meromorphic solutions of the functional equation

$$\varphi(u+v)+\varphi(u-v) = R(\varphi(u), \varphi(v)),$$

in which R is rational, are shown to be of the form $A\varphi(ku)+B$, where A, B, and k are constants and $\varphi(u)= u, u^2, 1/u^2, \cos u, 1/\cos u, \mathrm{cn}\, u$ (Jacobian elliptic function), $1/\mathrm{cn}\, u$, or $\wp(u)$ (Weierstrassian elliptic function). Furthermore, it is proved that the equation does not possess any other continuous solutions. *O. Lehto* (Helsinki)

SEQUENCES, SERIES, SUMMABILITY

See also 3771.

3898:

Benneton, Gaston. Sur les produits infinis complexes semi-convergents. Bull. Sci. Math. (2) **84** (1960), 7–10.

A detailed exposition of the results previously stated by the same author [C. R. Acad. Sci. Paris **247** (1958), 1284; MR **20** #5987]. *A. G. Azpeitia* (Providence, R.I.)

3899:

Billard, Pierre. Sur la presque convergence des suites. C. R. Acad. Sci. Paris **251** (1960), 618–619.

Let $S\sim\sum_1^\infty a(n)$ be a divergent series with $a(n)\geqq 0$ and $na(n)\to 0$. A sequence $\{n_i\}$ of integers is said to be rare-S if it is finite or $\sum_1^\infty a(n_i)<\infty$; it is said to be full-$S$ if its complement in the positive integers is rare-S. A sequence $\{s(n)\}$ is said to almost converge-S to s if there exists a full-S sequence $\{n_k\}$ such that $\{s(n_k)\}$ converges to s. The author also defines notions of the above type relative to convergence almost everywhere of sequences of functions. Results are stated relating the notions to convergence in measure, and applications are stated with respect to trigonometric series. *P. Civin* (Gainesville, Fla.)

3900:

Jurkat, W. B. Zur gliedweisen Differentiation fast überall. Arch. Math. **9** (1958), 347–354.

Let $\{s_n(x)\}$ be a sequence of complex-valued functions defined on the finite interval $[a, b]$. The author assumes the hypothesis (A): $s_n(x)\to s(x)$ (ordinary convergence) for all x, and there exists a function $t(x)$ such that, for each $n\geqq 0$ and for all x_0 and x_1 $(a\leqq x_0<x_1\leqq b)$, the inequality $|s_n(x_1)-s_n(x_0)|\leqq t(x_1)-t(x_0)$ holds. (For real-valued s_n, the requirement on t reduces to this, that each of the functions $t(x)\pm s_n(x)$ is nondecreasing in $[a, b]$.) The problem: to find conditions under which (A) implies, in as strong a sense as possible, the conclusion (B): $s_n'(x)\to s'(x)$.

Let S denote the space of functions measurable on $[a, b]$, with the Fréchet metric. In S, C_0^*-convergence denotes ordinary convergence (p.p.); and $f_n\to f$ (C_1^*-convergence) if, for each increasing sequence $\{n_k\}$,

$$(f_{n_1}(x)+f_{n_2}(x)+\cdots+f_{n_k}(x))/k\to f(x)\quad \text{(p.p.)}.$$

Also, $f_n\to f$ strongly [weakly] if each subsequence of $\{f_n\}$ has a subsequence which is C_0^*-convergent [C_1^*-convergent] to f.

In the space L^2 metrized by the inner product, $f_n\to f$ strongly [weakly] if $(f_n-f, f_n-f)\to 0$ [if $(f_n-f, g)\to 0$ for each g in L^2]. In L^2 and in S, a sequence is strongly [weakly] compact if each of its subsequences has a strongly [weakly] convergent subsequence.

Principal results: In S and in L, (A) implies (B) in the sense of weak convergence; if in addition $\{s_n'\}$ is strongly compact, then (A) implies (B) in the sense of strong convergence. If $\{s_n'\}$ converges p.p., then (A) implies (B) in the sense of convergence p.p. $f_n\to f$ weakly [strongly] in L^2 if and only if $\{f_n\}$ is weakly [strongly] compact in L^2 and either $f_n\to f$ weakly in S or else the limit of each C_1^*-convergent subsequence of f_n is f, almost everywhere.

The author lists some open questions. Examples: Can weak convergence in S be regarded as convergence in some Hausdorff topology? What happens to the concept of weak convergence in S if it is defined in terms of some regular summability method other than C_1? The list of references is impressive. *G. Piranian* (Ann Arbor, Mich.)

3901:

Agnew, Ralph Palmer. Riemann methods for evaluation of series. Tôhoku Math. J. (2) **11** (1959), 385–405.

If p is a positive integer, the series $\sum u_n$ with partial sums $\{s_n\}$ is evaluable $R(p)$ to L if the series in $\sigma_1(p, t)= \sum_{k=0}^\infty ((\sin kt)/kt)^p u_k$ converges over some interval $0<t<t_0$ and $\sigma_1(p, t)\to L$ as $t\to 0$. The series is evaluable R_p to L if the series $\sigma_2(p, t)=c_p t\sum_{k=0}^\infty ((\sin kt)/kt)^p s_k$ converges over some interval $0<t<t_0$ and $\sigma_2(p, t)\to L$ as $t\to 0$. c_p is defined so that $s_k\equiv 1$ implies $\sigma_2\to 1$ as $t\to 0$. $R(p)$ and R_p are not regular if $p=1$ but are regular if $p\geqq 2$. The purpose of this paper is to compare the methods $R(p)$ and R_p for series $\sum u_n$ satisfying the Tauberian condition (*) $\limsup |nu_n|\leqq M<\infty$. A least constant A depending upon p, and positive functions $n(\alpha)$, an integer, and $t(\alpha)$ such that $n(\alpha)\to\infty$ and $t(\alpha)\to 0$ as $\alpha\to\infty$, are determined such that

$$(^{**})\qquad \limsup_{\alpha\to\infty} |\sigma_1(p, t)-s_n| \leqq A \limsup_{n\to\infty} |nu_n|$$

for series satisfying (*). A similar result is obtained for $\sigma_2(p, t)$. It follows that if $\sum u_n$ is a series for which $nu_n\to 0$, then the $R(p)$ and R_p transforms are equiconvergent. Under the assumption that (*) holds, a relation similar to (**) is obtained for the difference $|\sigma_1(p, t)-\sigma_2(p, \lambda t)|$, λ a positive constant. The paper concludes with a discussion of product transformations $R(p)C_r$ and R_pC_r where C_r is the Cesàro method of order r. Let T denote either $R(p)$ or R_p. If T is such that $\sum u_n$ is evaluable T to L whenever $\sum u_n$ converges to L and $\lim nu_n=0$, and if $r>0$, then each series evaluable C_{r-1} to L is evaluable TC_r to L. *J. G. Herriot* (Stanford, Calif.)

3902:

Meyer-König, W.; Zeller, K. On Borel's method of summability. Proc. Amer. Math. Soc. **11** (1960), 307–314.

Die Verfasser behandeln verschiedene Fragen bei den Borel-Verfahren B (erklärt durch $b(x)=e^{-x}\sum s_n x^n/n! \to s$ $(x\to+\infty)$) und B_1 (erklärt durch $b(n)\to s$ $(n=1, 2, \cdots)$); dabei wird unterschieden zwischen regulärer und singulärer Summierbarkeit, je nachdem $f(z)=\sum a_n z^n$ einen Konvergenzradius $\rho>0$ oder $\rho=0$ hat. (1) Ist $\sum a_n$ regulär B-summierbar, so ist $f(z)$ in $D(\rho)=\{|z|<\rho\}\cup\{|z-\frac{1}{2}|<\frac{1}{2}\}$ regulär, und zu gegebenem ρ $(0<\rho<1)$ gibt es regulär B-summierbare Reihen $\sum a_n$, deren $f(z)$ genau in $D(\rho)$ regulär ist. Entsprechendes gilt für singuläre B-Summierbarkeit. (2) Ist $\sum a_n$ regulär B_1-summierbar, so ist $f(z)$ in $\{|z|<\rho\}\cup\{|z-c|<c\}$ regulär, wobei $c=\frac{1}{2}(\sigma^2-\pi^2)^{-1/2}$ falls $\sigma=\rho^{-1}\geqq(\pi^2+1)^{1/2}$, und sonst $c=\frac{1}{2}$ ist; diese Zahl c kann nicht vergrößert werden. Daraus folgt ein neuer Beweis eines Satzes des Ref. [Math. Z. **68** (1958), 488–497; MR **19**, 1170], daß die regulären B- und B_1-Verfahren für Fabrysche Lückenreihen äquivalent sind. (3) Die Wirkfelder $\mathfrak{B}$ und $\mathfrak{B}_1$ von B und B_1 sind keine BK-Räume, weshalb B, B_1 nicht äquivalent zu zeilenfiniten Matrixverfahren sind. (4) Für das B_1-Verfahren gibt es keinen reinen Lückensatz [Erdös, Bull. Acad. Polon. Sci. Cl. III **5** (1957), 105–107; MR **19**, 135]: Zu jeder Folge $\{k_m\}$ natürlicher Zahlen gibt es eine divergente, B_1-summierbare Reihe $\sum a_k$ mit $a_k=0$ $(k\neq k_m)$. Beweis mit einem Satz von Eidelheit-Pólya über die Lösbarkeit unendlicher Gleichungssysteme.

D. Gaier (Pasadena, Calif.)

3903:

Wlodarski, L. On a certain method of Toeplitz. Ann. Polon. Math. **7** (1959), 41–49.

The author shows that the Toeplitz matrix (a_{mn}) with

$$a_{mn}=\frac{(2e)^{-m}m^{n2^{-m}}}{\Gamma(n2^{-m}+1)}$$

is regular, and that it sums the sequence of partial sums of the series $\sum z^n$ to ∞ for all real z greater than or equal to 1, and to $1/(1-z)$ for all other complex values z.

G. Piranian (Ann Arbor, Mich.)

3904:

Sergušov, S. A. Application of Voronoi's regular methods to the Cauchy multiplication of series. Uspehi Mat. Nauk **15** (1960), no. 1 (91), 225–232. (Russian)

Let $p_0>0$, $p_n\geqq 0$, $P_n=p_0+\cdots+p_n$, $p_n\times q_n=p_0q_n+\cdots+p_nq_0$. By $\sum_{n=0}^\infty a_n=S(W, p_n)$ is meant that the series $\sum_{n=0}^\infty a_n$ is summable by Voronoi's regular method to the sum S, i.e., $\lim_{n\to\infty}(a_nP_0+\cdots+a_0P_n)/P_n=S$. The following theorem represents the main result of the author's note: If $\sum_{n=0}^\infty a_n=A(W, p_n)$, $\sum_{n=0}^\infty b_n=B(W, q_n)$ and $|b_n\times q_n|\times P_n=O(q_n\times P_n)$, then $\sum_{n=0}^\infty c_n=AB(W, p_n\times q_n)$, where $c_n=a_n\times b_n$. The theorems obtained by F. Mears [Bull. Amer. Math. Soc. **41** (1935), 875–880] follow from this theorem. There also is another theorem: If $a_n\times P_n=O(P_n)$ and $|b_n\times q_n|\times P_n=o(q_n\times P_n)$, then $\sum_{n=0}^\infty c_n=o(W, p_n\times q_n)$. A few corollaries follow from both theorems.

J. W. Andrushkiw (Newark, N.J.)

3905:

Srivastava, Pramila. Tauberian theorems for absolute summability of series and integrals. Math. Z. **73** (1960), 460–465.

The series $\sum a_n$ is summable $|A, \lambda|$, for an increasing sequence λ_n tending to ∞, if $\sum a_n e^{-\lambda_n x}$ converges to sum $f(x)$ for $x>0$ and if $f(x)$ has bounded variation for $x>0$. The series is summable $|R, \lambda, k|$ $(k\geqq 0)$ if $\omega^{-k}\sum_{\lambda_n<\omega}(\omega-\lambda_n)^k a_n$ has bounded variation for $\omega>0$.

The main result of the paper is that, if $\sum a_n$ is summable $|A, \lambda|$, then summability $|R, \lambda, k|$ is equivalent to the bounded variation of $\omega^{-(k+1)}\sum_{\lambda_n<\omega}(\omega-\lambda_n)^k\lambda_n a_n$. It is easy to deduce from the case $k=0$ that necessary and sufficient conditions for the absolute convergence of $\sum a_n$ are summability $|A, \lambda|$ and the bounded variation of $\omega^{-1}\sum_{\lambda_n<\omega}\lambda_n a_n$. The results generalise easily from series to Stieltjes integrals. They give analogues for absolute summability $|A, \lambda|$ and $|R, \lambda, k|$ of theorems for ordinary summability (A, λ) and (R, λ, k) due to Rajagopal [Amer. J. Math. **69** (1947), 371–378, 851–852; MR **9**, 26, 278] and Widder [*The Laplace transform*, Princeton Univ. Press, Princeton, N.J., 1941; MR **3**, 232]; and extend a theorem of Hyslop [J. London Math. Soc. **12** (1937), 176–180].

H. R. Pitt (Nottingham)

3906:

Butzer, P. L. Tauberian conditions: a remark on a paper by C. T. Rajagopal. Arch. Math. **8** (1957), 405–408.

The author raises the following very general question. Let P and Q be two summation methods, P weaker than Q. Are there non-trivial Tauberian conditions for the P-method which are not Tauberian for the Q-method? He cites particular cases in which an affirmative answer has been given and mentions a number of other cases in which examples would be of great interest.

D. Waterman (Lafayette, Ind.)

3907:

Pati, T. Tauberian theorems for absolute Riesz summability. Indian J. Math. **1**, 61–68 (1959).

Given a series $\sum a_n$ and a monotone increasing sequence of positive numbers $\{\lambda_n\}$, tending to infinity with n. Set $A_\lambda(\omega)=\sum_{\lambda_n\leqq\omega}a_n$, $A_\lambda{}^r(\omega)=\int_0^\omega(\omega-\tau)^r dA_\lambda(\tau)$. The series $\sum a_n$ is said to be summable $|R, \lambda, r|$, $r\geqq 0$, if $\omega^{-r}A_\lambda{}^r(\omega)\in \mathrm{BV}(a, \infty)$, $a>0$. Set $\mu_1=\lambda_1$, $\mu_n=\lambda_n-\lambda_{n-1}$, $c_n=a_n\lambda_n/\mu_n$, $D(t)=\sum_{\lambda_n\leqq t}(c_n-c_{n+1})$, and let $F(\omega)=\lambda_n$ for $\lambda_n\leqq\omega<\lambda_{n+1}$. A necessary and sufficient condition that $\sum a_n$ be absolutely convergent when it is summable $|R, \lambda, 1|$ and $D(\omega)\in\mathrm{BV}(a, \infty)$, is that $[D(\omega)-a_1]F(\omega)/\omega\in\mathrm{BV}(a, \infty)$. A similar but more complicated condition permits the passage from summability $|R, \lambda, 2|$ to summability $|R, \lambda, 1|$ when the sequence $\{c_n\}$ is summable $|R, \lambda, 1|$.

E. Hille (New Haven, Conn.)

3908:

Mazhar, Syed Mohammad. A Tauberian theorem for absolute summability. Indian J. Math. **1**, 69–76 (1959).

The notions of absolute summability used in this note were introduced by T. M. Flett [Proc. London Math. Soc. (3) **7** (1957), 113–141; MR **19**, 266]. Summability $|A|_k$ and $|C, \alpha|_k$ are generalizations of absolute Abel and Cesàro summability, to which they reduce for $k=1$. J. M. Hyslop [J. London Math. Soc. **12** (1937), 176–180] proved that if $\sum a_n$ is summable $|A|$ and if the sequence $\{na_n\}$ is evaluable $|C, \alpha+1|$, $\alpha\geqq 0$, then $\sum a_n$ is summable $|C, \alpha|$. The author proves the corresponding theorem for $|A|_k$ and $|C, \alpha|_k$ summability where he can allow $\alpha>-1$.

E. Hille (New Haven, Conn.)

APPROXIMATIONS AND EXPANSIONS

See also 3780, 3830.

3909:

Stanku, D. D. [Stancu, D. D.] Some Taylor developments for functions of several variables. Rev. Math. Pures Appl. **4** (1959), 249–265. (Russian)

The author points out that many Taylor-type expansions of functions of several variables can be obtained as limiting cases of interpolation formulae. An example is the formula, due to Nicolescu,

$$\phi(x, y) = \sum_{j=0}^{n} D^{j-n}[D^{n-j}\phi(a, y) + D^{n-j}\phi(x, b) - D^{n-j}\phi(a, b)] + \frac{(x-a)^{n+1}(y-b)^{n+1}}{((n+1)!)^2} D^{n+1}\phi(\xi, \eta),$$

where $Df(x, y) = \lim_{h\to 0; k\to 0} [f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x, y)]/hk$ and $D^{-1}f(x, y) = \int_a^x \int_b^y f(u, v)\,du\,dv$.

W. H. J. Fuchs (Ithaca, N.Y.)

3910:

Szász, Paul. On quasi-Hermite-Fejér interpolation. Acta Math. Acad. Sci. Hungar. **10** (1959), 413–439. (Russian summary, unbound insert)

The author generalizes certain important concepts introduced by Fejér, like step parabolas, quasi-normal sets of abscissas, etc. The nature of the results is illustrated by the following theorem which contains a previous one due to Egerváry and Turán [same Acta **9** (1958), 259–267; MR **21** #2136]. Let $f(x)$ be continuous in $-1 \le x \le 1$ and let the polynomials $S_n(x)$ of degree $2n+1$ be defined by the following conditions: $S_n(\pm 1) = f(\pm 1)$, $S_n(x_{n\nu}) = f(x_{n\nu})$, $S_n'(x_{n\nu}) = y_{n\nu}'$, $1 \le \nu \le n$, where $x_{n\nu}$ are the zeros of the Legendre polynomial $P_n(x)$ and $y_{n\nu}'$ given constants. If $|y_{n\nu}'| \le \Delta$, Δ independent of ν and n, then $S_n(x) \to f(x)$ uniformly in $-1 \le x \le 1$. This holds also for the zeros of the Chebychev polynomials of the first kind as abscissas. Quasi-normal abscissas are defined in a manner analogous to that of Fejér, only the points $x = \pm 1$ are added as interpolation points for S_n (but not for S_n'). Generalizing a result of Fejér, the uniform convergence of the Lagrange polynomials associated with quasi-normal abscissas is proved, provided that $f(x)$ satisfies a Lipschitz condition of exponent $> \frac{1}{2}$. The case of Jacobi abscissas is indicated. Finally the concept of strongly quasi-normal abscissas is defined.

G. Szegö (Stanford, Calif.)

3911:

Kiš, O. [Kis, Ottó]. Notes on interpolation. Acta Math. Acad. Sci. Hungar. **11** (1960), 49–64. (Russian)

The author is concerned with (0, 2) interpolation by polynomials (à la Turán): given nodes x_k and values α_k, β_k, the (0, 2) interpolating polynomial R_n of degree $2n-1$, if it exists, has $R_n(x_k) = \alpha_k$, $R_n''(x_k) = \beta_k$. In this paper he takes the nodes to be $\exp(2k\pi i/n)$ $(k = 1, 2, \cdots, n)$; then for $n \ge 2$ the interpolating polynomial always exists, and the author finds it explicitly. He also investigates the convergence of $R_n(z)$ with $\alpha_k = f(x_k)$, showing that if $\beta_k = o(n^2/\log n)$ $(k = 1, 2, \cdots, n)$ and $f(z)$ is regular for $|z| < 1$, continuous for $|z| \le 1$, with the modulus of continuity ω of $f(\exp ix)$ satisfying $\omega(\delta) \log \delta \to 0$, then $R_n(z) \to f(z)$ uniformly in $|z| \le 1$. He also finds the $(0, 1, \cdots, r-2, r)$ interpolating polynomials explicitly and gives a similar convergence theorem for (0, 1, 3) interpolation.

R. P. Boas, Jr. (Evanston, Ill.)

3912:

Erdős, P. On random interpolation. J. Austral. Math. Soc. **1** (1959/61), 129–133.

The function $L_n(t, \theta)$ is defined as the unique trigonometrical polynomial in θ of degree not exceeding n for which $L_n(t, \alpha_\nu) = \phi_\nu(t)$, where $\phi_\nu(t)$ is the νth Rademacher function and $\alpha_\nu = 2\pi\nu(2n+1)^{-1}$ $(\nu = 0, 1, \cdots, 2n)$. If $M_n(t) = \max |L_n(t, \theta)|$ for $0 \le \theta \le 2\pi$, it is proved that there is an absolute constant C such that $|M_n(t) - 2\pi^{-1} \log \log n - C| \le \eta_n$ outside a set (depending on n) of measure μ_n, where μ_n, η_n can be defined to satisfy either (a) $\mu_n \le n^{-c_1}$, $\eta(n) \le c_2(c_1)$ for some positive number c_2 defined for each positive c_1, or (b) $\mu_n \to 0$, $\eta_n \to 0$. The result (a) is proved and improves the result $M_n(t) \le [2 + o(1)](\log n)^{1/2}$ of Salem and Zygmund [Proc. Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, Vol. II, 243–246, Univ. of Calif. Press, Berkeley and Los Angeles, 1956; MR **18**, 891]. It is also stated, following work of Salem and Zygmund [Acta Math. **91** (1954), 245–301; MR **16**, 467], that there is a distribution function $\psi(\alpha)$ so that, neglecting a set in t whose measure tends to 0 as $n \to \infty$, the measure of the θ-set in which $L_n(t, \theta) < \alpha$ tends to $\psi(\alpha)$.

H. R. Pitt (Nottingham)

3913:

van Rossum, H. Contiguous orthogonal systems. Nederl. Akad. Wetensch. Proc. Ser. A **63** = Indag. Math. **22** (1960), 323–332.

Let $\gamma = \{c_0, c_1, c_2, \cdots\}$ be a sequence of real numbers and Ω the linear operator working on polynomials in x such that $\Omega(x^n) = c_n$ $(n = 0, 1, 2, \cdots)$. If the sequence is semi-normal, that is, if certain determinants in the c's are different from zero, it is known that there exists a uniquely determined system of monic polynomials $B_n(x) = x^n + b_{n,1}x^{n-1} + \cdots + b_{n,n}$ $(n = 0, 1, 2, \cdots)$, which is orthogonal with respect to the sequence, i.e., $\Omega\{B_n(x) \cdot B_m(x)\} = 0$ if $n \ne m$; $\ne 0$ if $n = m$. If γ is semi-normal,

$$\gamma_k = \{c_k, c_{k+1}, c_{k+2}, \cdots\}$$

is semi-normal. Let $\{B_n^{(k)}(x)\}$ be the system of monic orthogonal polynomials orthogonal with respect to γ_k. A number of results are found. The principal one is that, for any $k = 0, 1, 2$, the three polynomials $B_n^{(k)}(x)$, $B_n^{(k+1)}(x)$, $B_{n-1}^{(k+1)}(x)$ are linearly dependent, and the exact dependence relation is shown. (The systems $\{B_n^{(k)}(x)\}$ and $\{B_n^{(k+1)}(x)\}$ are called contiguous orthogonal systems of polynomials.)

E. Frank (Chicago, Ill.)

3914:

Tandori, Károly. Bemerkung zu einem Satz von G. Alexits. Acta Sci. Math. Szeged **21** (1960), 12–14.

The author proves that an earlier result of Alexits [same Acta **16** (1955), 127–129; MR **17**, 843], on the "sehr stark" (Cesàro) summability of the Abel summable orthogonal series $\sum a_n\phi_n(x)$ with $a_n = O(c_n^2)$, remains valid even without the assumption on the increasing nature of $\{n^2c_n^2\}$.

M. S. Ramanujan (Ann Arbor, Mich.)

3915:

Tandori, Károly. Ein Summationssatz für Orthogonalreihen mit monotoner Koeffizientenfolge. Acta Sci. Math. Szeged **21** (1960), 15–18.

Let $\{\phi_n(x)\}$ be an orthonormal system in $[a, b]$ and let $\{a_n\}$ be a positive monotone decreasing sequence. The author proves that if there exists an increasing sequence $\{n_k\}$ of indices such that

$$\sum_{k=1}^{\infty} \min\{a_{n_k}\sqrt{(n_{k+1}-n_k)},\ a_{n_k}^2(n_{k+1}-n_k)\log^2 k\} < \infty,$$

then the orthogonal series $\sum_1^\infty a_n\phi_n(x)$ is $(C, 1)$-summable p.p. in $[a, b]$. *M. S. Ramanujan* (Ann Arbor, Mich.)

3916:

de Leeuw, Karel. On the degree of approximation by Bernstein polynomials. J. Analyse Math. **7** (1959), 89–104.

The author determines the saturation class of the Bernstein polynomials $B_n(f, x)$ [see G. G. Lorentz, *Bernstein polynomials*, Univ. of Toronto Press, Toronto, 1953; MR **15**, 217], i.e., the class of all continuous functions $f(x)$ which have the best possible degree of approximation by the $B_n(f, x)$. He proves the following theorem. In order that $|B_n(f, x)-f(x)| = O(n^{-1})$ should hold uniformly on each interior subinterval (c, d) of $[a, b]$ $(0 \leq a < c < d < b \leq 1)$, it is necessary and sufficient that there exist on $[a, b]$ a function $h(t)$, bounded on each interval (c, d), such that $f(x)$ and $\int_c^x dt \int_c^t h(u)du$ differ by a linear function on (c, d). If in addition $|B_n(f, x)-f(x)| = o(n^{-1})$ a.e. on $[a, b]$, then $f(x)$ is linear. The main difficulty is to obtain the necessity of the condition. For this purpose the $B_n(f, x)$ are replaced by similar polynomials $A_n(f, x)$, which contain certain integral averages instead of the values $f(k/n)$. The main tools of the proof are the conjugate operators $A_n^*(f)$ defined by $\langle A_n^*(f), g\rangle = \langle f, A_n(g)\rangle = \int_0^1 fA_n(g)dx$; and the property of the A_n^* that

$$\langle n(A_n^*(\varphi)-\varphi), f\rangle \to \langle (q\varphi)'', f\rangle,$$

$q = \frac{1}{2}x(1-x)$, if φ is twice differentiable. *G. G. Lorentz* (Syracuse, N.Y.)

3917:

Berg, Lothar. Verallgemeinerungen des Kriteriums von Herrn H. Schubert. Math. Nachr. **20** (1959), 159–165.

Complicated but quite general conditions on $g(s, t)$ and $x = x(s)$ are given under which the asymptotic relation

$$(2\pi)^{-1/2}\int_a^b \exp[-g(s, t)]dt \sim \exp[-g(s, x)]\cdot[\partial^2 g(s, x)/\partial t^2]^{-1} \quad (s \to S)$$

holds. Application is made to integrals of the form $\int_0^\infty \exp[-ct^a + \chi(t^a)]dt$ which a previous result of H. Schubert could not handle. *R. R. Goldberg* (Evanston, Ill.)

3918:

Berg, Lothar. Asymptotische Darstellungen für verallgemeinerte Fourierintegrale. Math. Nachr. **20** (1959), 166–170.

Formulas such as

$$\int_a^b \cos g(s, t)\, dt = \{2\pi[\partial^2 g(s, x)/\partial x^2]^{-1}\}^{1/2}\{\cos[g(s, x)+\pi/4]+o(1)\} \quad (s \to S)$$

are developed for suitable functions $g(s, t)$ and $x(s)$. The known formula

$$J_n(s) = (2/\pi s)^{1/2}\{\cos(s-\tfrac{1}{2}n\pi-\tfrac{1}{4}\pi)+o(1)\} \quad (s \to \infty)$$

is a special case. *R. R. Goldberg* (Evanston, Ill.)

FOURIER ANALYSIS

3919:

Erdös, P. About an estimation problem of Zahorski. Colloq. Math. **7** (1959/60), 167–170.

The problem is to estimate $I = \int_0^{2\pi} |\sum_{k=1}^n \cos n_k x|dx$, where the n_k are integers; an upper bound $cn^{1/2}$ is trivial. The author shows in an elementary way that for $n_k = k^2$ we have $I > cn^{1/2-\varepsilon}$. He also shows that there is a sequence $\{n_k\}$ such that $I = (\pi n_k)^{1/2} + o(n_k^{1/2})$; the proof depends on results of Salem and Zygmund [Acta Math. **91** (1954), 245–301; MR **16**, 467]. *R. P. Boas, Jr.* (Evanston, Ill.)

3920:

Kadec, M. I. On the distribution of points of maximum deviation in the approximation of continuous functions by polynomials. Uspehi Mat. Nauk **15** (1960), no. 1 (91), 199–202. (Russian)

Let $f(x)$ be a continuous function on $[0, \pi]$, $T_n(x)$ the even trigonometric polynomial of best approximation, $x_k^{(n)}$ the points where $f(x)-T_n(x)$ takes the values $\max_x |f(x)-T_n(x)|$ with alternating signs. The author proves that $\Delta_n = \max_k |x_k^{(n)} - \pi k/(n+1)|$ has the property $\liminf (n^{1/2-\varepsilon}\Delta_n) = 0$ for each $\varepsilon > 0$. *G. G. Lorentz* (Syracuse, N.Y.)

3921:

Garkavi, A. L. Simultaneous approximation to a periodic function and its derivatives by trigonometric polynomials. Izv. Akad. Nauk SSSR. Ser. Mat. **24** (1960), 103–128. (Russian)

Let $W^{(r)}$ denote the set of 2π-periodic functions having bounded rth derivative. For $f \in W^{(r)}$ and $s = 0, 1, \cdots, r$, let $T_n(f^{(s)})$ be the trigonometric polynomial (t.p.) of degree n which best approximates $f^{(s)}$, that is,

$$E_n(f) \equiv \inf_{T_n} \|f^{(s)} - T_n\| = \|f^{(s)} - T_n(f^{(s)})\|,$$

the inf being taken over all t.p. of degree n. Here $\|\cdot\|$ denotes the sup norm. For some f it happens that the polynomials of best approximation for the derivatives of f are not the derivatives of the polynomial of best approximation for f. In this case, for any t.p. T_n it is thus true that $\|f^{(s)} - T_n^{(s)}\| > E_n(f^{(s)})$ for one or more values of s. In this connection the problem arises of evaluating $C_{n,r}$, where $C_{n,r} = \sup_{f\in W^{(r)}} C_{n,r}(f)$ and

$$C_{n,r}(f) = \inf_{T_n} \max_{0\leq s\leq r} \|f^{(s)} - T_n^{(s)}\|\cdot[E_n(f^{(s)})]^{-1}.$$

(The T_n for which this inf is attained is not in general unique.) The magnitude of $C_{n,r}(f)$ shows how well, in

comparison with best approximation for f, it is possible to simultaneously approximate f and its first r derivatives by an nth order t.p. and its first r derivatives. The author obtains the formula $C_{n,r}=4\pi^{-2}\ln(p+1)+O(\ln\ln\ln p)$ where $p=\min(n,r)$. Also, for any $f\in W^{(r)}$ and any t.p. T_n, it is shown that

$$\|f^{(r)}-T_n^{(r)}\| \leqq \|f-T_n\|n^r+C_{n,r}\{n^rE_n(f)+E_n(f^{(r)})\} \leqq \|f-T_n\|n^r+(1+\pi/2)C_{n,r}E_n(f^{(r)}).$$

In the first of these inequalities the $C_{n,r}$ is asymptotically best possible. *R. R. Goldberg* (Evanston, Ill.)

3922:

Timan, A. F. On the question of simultaneous approximation of functions and their derivatives on the whole real axis. Izv. Akad. Nauk SSSR. Ser. Mat. **24** (1960), 421–430. (Russian)

For every bounded f uniformly continuous on $(-\infty,\infty)$, and every $\sigma\geqq 0$, there exists an entire function $g_\sigma(f)$ of type $\leqq\sigma$ which, in the class of all such functions g_σ, gives the best uniform approximation on $(-\infty,\infty)$ to f; that is, $A_\sigma(f)\equiv\inf_{g_\sigma}\|f-g_\sigma\|=\|f-g_\sigma(f)\|$, where $\|\cdot\|$ denotes the sup norm. Generalizing a theorem of Bernstein [C. R. (Doklady) Acad. Sci. URSS (N.S.) **51** (1946), 331–334, 487–490; MR **8**, 20], the author proves the following theorem. A function bounded on $(-\infty,\infty)$ has a bounded and uniformly continuous rth order derivative if and only if $A_\sigma(f)\to 0$ as $\sigma\to\infty$ and

$$\lim_{\min(\sigma,\tau)\to\infty}\|g_\sigma^{(r)}(f)-g_\tau^{(r)}(f)\| = 0.$$

In this case $\|f^{(r)}-g_\sigma^{(r)}(f)\|=O[A_\sigma(f^{(r)})]$. Corresponding to Garkavi's result on 2π-periodic functions [see preceding review] the following formula is established: $C_{\sigma,r}=4\pi^{-2}\ln(r+1)+O(\ln\ln\ln r)$ (as $r\to\infty$) uniformly with respect to $\sigma>0$. Here

$$C_{\sigma,r} = \sup\inf_{g_\sigma}\max_{0\leqq k\leqq r}\|f^{(k)}-g_\sigma^{(k)}\|\cdot[A_\sigma(f^{(k)})]^{-1},$$

where g_σ ranges over all entire functions of type $\leqq\sigma$ and the sup is taken over all bounded f on $(-\infty,\infty)$ with bounded rth derivative. *R. R. Goldberg* (Evanston, Ill.)

3923:

Ivašev-Musatov, O. S. On the coefficients of trigonometric null-series. Amer. Math. Soc. Transl. (2) **14** (1960), 289–310.

Translation of Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 559–578 [MR **20** #5392].

3924:

Bari, N. K. Subsequences converging to zero everywhere of partial sums of trigonometric series. Izv. Akad. Nauk SSSR. Ser. Mat. **24** (1960), 531–548. (Russian)

V. Ya. Kozlov [Mat. Sb. (N.S.) **26** (88) (1950), 351–364; MR **12**, 174] proved that there are non-trivial sine series $\sum b_n\sin nx$ for which a sub-sequence $\{S_{n_k}(x)\}$ of the partial sums converges to 0 everywhere; moreover the convergence is uniform in $0<\delta\leqq x\leqq\pi-\delta$ for every position $\delta<\frac{1}{2}\pi$.

It is shown in this paper that it is possible to give an example of such a series for which the indices n_k of the sub-sequence satisfy $n_{k+1}/n_k=O(g(k))$, where $g(k)$ is any function increasing monotonically to ∞ with k.

W. H. J. Fuchs (Ithaca, N.Y.)

3925:

Men'šov, D. E. On convergent sequences of partial sums of a trigonometric series. Mat. Sb. (N.S.) **48 (90)** (1959), 397–428. (Russian)

Der Autor beweist hier in teilweise verallgemeinerter Form seine früher aufgestellten Sätze über die Funktionenmenge der Grenzwerte von Teilsummen trigonometrischer Reihen bei Betrachtung der Konvergenz fast überall [Dokl. Akad. Nauk SSSR **106** (1956), 777–780; MR **17**, 1080]. Daneben enthält die Arbeit die unten angeführten Sätze. Definitionen: Seien $u_n(x)$ $(n=1,2,\cdots)$ meßbare Funktionen, endlich fast überall in $[a,b]$. Dann ist die Grenzfunktion $\varphi(x,E)$ von (1) $\sum_{n=0}^\infty u_n(x)$ in der Menge $E\subset[a,b]$ maximal, wenn keine andere Grenzfunktion $f(x,E')$ von (1) mit $E\subset E'\subset[a,b]$, mes $(E'-E)>0$, $f(x,E')=\varphi(x,E)$ für fast alle $x\in E$ existiert. Die ebene Punktmenge $K=\{(x,y)\}$ ((x,∞) und $(x,-\infty)$ gegebenenfalls inbegriffen) heißt abgeschlossen durch Senkrechte, wenn ihr Durchschnitt K_x mit einer beliebigen Geraden $x=x_0$ eine lineare, abgeschlossene Menge ist. (K_x kann $y=+\infty$ oder $y=-\infty$ enthalten.) Satz 5: Sei K eine ebene durch Senkrechte abgeschlossene Menge und falle die Projektion von K auf die x-Achse zusammen mit $[-\pi,\pi]$. Seien meßbare Funktionen $F(x)$, $G(x)$ fast überall definiert in $[-\pi,\pi]$, und für fast alle $x\in[-\pi,\pi]$ und alle y für die $(x,y)\in K$ ist, sei $G(x)\leqq y\leqq F(x)$. Ist dann $M=\{\varphi(x)\}$ die Menge aller meßbaren, in $[-\pi,\pi]$ fast überall definierten Funktionen für die $(x,\varphi(x))\in K$ für fast alle $x\in[-\pi,\pi]$, dann existiert eine Reihe (2) $\frac{1}{2}a_0+\sum_{n=0}^\infty (a_n\cos nx+b_n\sin nx)$, erfüllend die Bedingungen: (i) M ist die Menge aller Grenzfunktionen von (2) in $[-\pi,\pi]$; (ii) wenn irgendeine der Folgen der Teilsummen $S_{n_k}(x)$ $(k=1,2,\cdots)$ von (2) gegen $f(x)$ konvergiert auf einer gewissen Menge $e\subset[-\pi,\pi]$ mit mes $(e)>0$, dann existiert eine solche Folge von Funktionen $\varphi_n(x)\in M$ $(n=1,2,\cdots)$, daß $\lim_{n\to\infty}\varphi_n(x)=f(x)$ fast überall auf e; (iii) $F(x)$ und $G(x)$ stellen die oberen und unteren Limites dem Maß nach in $[-\pi,\pi]$ von (2) dar; (iv) $\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n=0$. Satz 6: Wenn $\varphi(x,E)$ die Grenzfunktion von (1) in der Menge E ist, dann existiert eine maximale Grenzfunktion $\chi(x,H)$ dieser Reihe auf einer gewissen Menge H mit $E\subset H$, $\chi(x,H)=\varphi(x,E)$ fast überall auf E. [Vgl. auch die Arbeiten, Trudy Mat. Inst. Steklov. **32** (1950) = Amer. Math. Soc. Transl. No. 105 (1954); Moskov. Gos. Univ. Uč. Zap. **165**, Mat. **7** (1954), 3–33; Dokl. Akad. Nauk SSSR **114** (1957), 476–478; Trudy Moskov. Mat. Obšč. **7** (1958), 291–334; MR **12**, 254; **15**, 866; **16**, 467; **20** #1156; **21** #2152].

G. Goes (Evanston, Ill.)

3926:

Darsow, W. On boundedness of trigonometric series. J. London Math. Soc. **35** (1960), 237–238.

Let

$$s_n(x) = \tfrac{1}{2}a_0+\sum_{k=1}^n (a_k\cos kx+b_k\sin kx)$$

denote the nth partial sum of a trigonometric series. It is shown that if $s_n(x)$ is bounded below for each x in a set E of positive measure then $s_n(x)$ is also bounded above for almost every x in E. The principal step in the proof is

showing the existence of a constant $1<c<2$ such that $s_n(x)=O(n^c)$ for almost every x in E. The result then follows easily from a consideration of the Abel means and the $(C, 1)$ means of the series.

J. G. Herriot (Stanford, Calif.)

3927:

Wintner, Aurel. Fourier constants and equidistant Riemann sums. J. Math. Pures Appl. (9) **36** (1957), 251–261.

Let f be a function defined and finite everywhere in $(-\infty, \infty)$ with period one and having a finite Lebesgue integral in $[0, 1]$, and let $\sum_{-\infty}^{\infty} c_n e(nx)$, $e(y)=\exp(2\pi iy)$, be the Fourier series of f, where we suppose that $c_0=0$. Then the Fourier series can be written as $f\sim\sum_1^{\infty} g_m(x)$ where $g_m(x)=c_m e(mx)+c_{-m}e(-mx)$ and $c_k=\int_0^1 fe(-kx)dx$. If we write $f_n(x)=(1/n)\sum_{m=1}^{\infty} f(x+m/n)$, then under the given hypothesis on f it follows that $f_n(x)\sim\sum_{m=1}^{\infty} g_{nm}(x)$. If the Fourier series of f converges to f everywhere then that of f_n will converge to f_n everywhere. Hence formally one has $g_n(x)=\sum_{m=1}^{\infty}\mu(m)f_{nm}(x)$ where $\mu(m)$ is the Möbius function. The author investigates, in the paper under review, the convergence of the last formula, which represents the terms of the Fourier series of f in terms of the equidistant Riemann sums f_n of f. Improving upon previous known results, it is shown that the series converges for every x if f satisfies a Lipschitz condition of order greater than $\frac{1}{2}$, and that it need not always converge even if the Fourier series converges absolutely. The author obtains an interesting condition for the analyticity of f in $[0, 1]$. He shows that the condition $|f_n(x)-\int_0^1 f(t)dt|\leq Kq^n$ for every n and real x and for some constants K and $q<1$ is necessary and sufficient for analyticity.

V. Ganapathy Iyer (Zbl **79**, 94)

3928:

Boyer, B. J. On the summability of derived Fourier series. Pacific J. Math. **10** (1960), 475–485.

L. S. Bosanquet [Proc. London Math. Soc. (2) **46** (1940), 270–289; MR **1**, 329] proved that the $(C, \alpha+r)$, $\alpha\geq 0$, summability of the rth derived Fourier series of a Lebesgue integrable function is equivalent to the (C, α) summability of the Fourier series of another appropriately defined function integrable in the Cesàro-Lebesgue sense. The author follows his method and gives a necessary and sufficient condition which ensures the generalized summability (α, β) given by Bosanquet and Linfoot [Quart. J. Math. Oxford Ser. **2** (1931), 207–229] of the rth derived Fourier series of a Cesàro-Perron integrable function.

G. Sunouchi (Evanston, Ill.)

3929:

Yano, Kenji. On a method of Cesàro summation for Fourier series. Kōdai Math. Sem. Rep. **9** (1957), 49–58.

Let $f(t)$ be a Lebesgue integrable function with period 2π, and

$$\mathfrak{S}[f] = \tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt),$$

$$\varphi(t) = f(x+t)+f(x-t)-2f(x),$$

$$\Phi_\alpha(t) = (\Gamma(\alpha))^{-1}\int_0^t (t-u)^{\alpha-1}\varphi(u)du \quad (\alpha>0).$$

Then the author shows: if $0<\beta<\gamma$, $\beta\leq\gamma-\beta+1$ and $\Phi_\beta(t)=O(t_\beta)$, then $\mathfrak{S}[f]$ is summable (C, α) to $f(x)$ at $t=x$, where $\alpha=\beta/(\gamma-\beta+1)$. In the proof of this theorem the author uses a new kernel which gives Cesàro mean of $\mathfrak{S}[f]$.

G. Sunouchi (Zbl **78**, 55)

3930:

Yano, Kenji. On Fejér kernels. Proc. Japan Acad. **35** (1959), 59–64.

From the author's papers already published [J. Nara Women's Univ. **1** (1951), 1–24; Kōdai Math. Sem. Rep. **9** (1957), 1–11, 49–58; MR **19**, 412; preceding review] those parts related to the Fejér kernels and their conjugates are selected and improved.

S. Ikehara (Tokyo)

3931:

Kumari, Sulaxana. On the logarithmic summability of the successively derived series of a Fourier series and of its conjugate series. Indian J. Math. **1**, 87–101 (1959).

The author generalizes a theorem of G. Sunouchi [Tôhoku Math. J. (2) **3** (1951), 71–88; MR **12**, 696] on the logarithmic summability of the derived Fourier series to the case of the rth derived series. $\sum a_n$ is (R, k) summable to sum s if $R_k(w)=(\log w)^{-k}\sum_{n<w}(\log w/n)^k a_n$, $k\geq 0$, tends to s as $w\to\infty$. Let $f(\theta)$ be a function of class L and period 2π. $P(t)$ is a polynomial of the rth degree and

$$g(t) = \frac{1}{2t^r}[\{f(x+t)-P(t)\}+(-1)^r\{f(x-t)-P(-t)\}].$$

It is shown that if the (R, δ) mean, $\delta\geq 0$, of the Fourier series of $g(t)$, at $t=0$, is of order $o\{(\log w)^r\}$, and if the series is $(R, r+\delta)$ summable to s, then the rth formal derivative of the Fourier series of f is summable $(R, r+\delta)$ at $\theta=x$ to sum $r!\,s$. A similar result holds for the rth derived conjugate series; the function g is replaced by

$$h(t) = \frac{1}{2t^r}[\{f(x+t)-P(t)\}-(-1)^r\{f(x-t)-P(-t)\}].$$

D. Waterman (Lafayette, Ind.)

3932:

Rath, P. C. The harmonic summation of the derived Fourier series. Duke Math. J. **25** (1957), 125–130.

The author shows the following theorem: If

$$g(t) \equiv \{f(x+t)-f(x-t)\}/4\sin\tfrac{1}{2}t-C$$

is of bounded variation in $(0, \pi)$, and $g(t)\to 0$ as $t\to 0$, then the differentiated series of the Fourier series of $f(x)$ is harmonic summable (that is, $(N, 1/n)$ summable) to the value C at the point x.

G. Sunouchi (Zbl **80**, 50)

3933:

Flett, T. M. Some theorems on odd and even functions. Proc. London Math. Soc. (3) **8** (1958), 135–148.

Let U be periodic with period 2π and let V be the conjugate of U, i.e.,

$$V(y) = -(2\pi)^{-1}\int_{-\pi}^{\pi} U(x)\cot\tfrac{1}{2}(x-y)dx,$$

the integral being a principal value. For U odd, $p>1$ and $-1/p<\alpha<2-1/p$, it is proved that

$$\int_{-\pi}^{\pi}|x|^{p\alpha}|V|^p dx \leq A(p, \alpha)\int_{-\pi}^{\pi}|x|^{p\alpha}|U|^p dx.$$

For general U and $-1/p<\alpha<1-1/p$ the same was stated by Babenko [Dokl. Akad. Nauk SSSR **62** (1948), 157–160; MR **10**, 249]. A limiting form of these theorems when $p=1$ is proved. If $p\geqq 1$, U is even and of bounded variation, then

$$\left(\int_0^\pi x^{-1}|V|^p dx\right)^{1/p} \leqq A(p)\int_0^\pi |dU|.$$

An analogous theorem with fractional integrals of U is also given. Finally the author gives applications to some theorems on Fourier coefficients.

A. Nordlander (Zbl **82**, 281)

3934:

Weiss, M.; Zygmund, A. On the existence of conjugate functions of higher order. Fund. Math. **48** (1959/60), 175–187.

In this paper certain extensions of the notion of conjugate function are investigated. A function $f(x)$ defined in a neighborhood of a point x_0 is said to have a generalized derivative of order r ($r=1, 2, \cdots$) if $f(x_0+t)=\alpha_0+\alpha_1 t+\cdots+\alpha_r t^r/r!+o(t^r)$ for $t\to 0$, the α_j denoting constants. The number α_r is called the rth generalized derivative of f at x_0 and is denoted by $f_{(r)}(x_0)$. In case f has a generalized derivative $f_{(r-1)}(x_0)$ we define $\delta_r(x_0, t)$ by the formula

$$[f(x_0+t)+f(x_0-t)]/2 = \alpha_0+\alpha_2 t^2/2!+\cdots+\alpha_{r-1}t^{r-1}/(r-1)!+\delta_r(x_0, t)t^r/(2r!)$$

if r is odd and by

$$[f(x_0+t)-f(x_0-t)]/2 = \alpha_1+\alpha_3 t^3/3!+\cdots+\alpha_{r-1}t^{r-1}/(r-1)!+\delta_r(x_0, t)t^r/(2r!)$$

if r is even.

It is well known that if f is periodic and integrable then the conjugate function

$$\tilde f(x) = -\pi^{-1}\int_0^\pi [f(x+t)-f(x-t)](2\tan t/2)^{-1}dt$$

$$= -\pi^{-1}\int_0^\infty [f(x+t)-f(x-t)]t^{-1}dt$$

exists almost everywhere. We define the conjugate of order r of f by $\tilde f_r(x)=-\pi^{-1}\int_0^\infty \delta_r(x, t)t^{-1}dt$. Thus the ordinary conjugate function is the conjugate of order 0 of f. Also if f is an rth integral, then $\tilde f_r(x)$ is the ordinary conjugate function of $f^{(r)}(x)$.

It is well known [A. Zygmund, *Trigonometric series*, 2nd ed., Vol. II, Cambridge, New York, 1959; MR **21** #6498; Chapter XI, § 5] that if f has a generalized rth derivative in a set E, then $\tilde f_r(x)$ exists almost everywhere in E. The principal theorem of the paper generalizes this result and is as follows: Let $f(x)$ be periodic and integrable and suppose that $f_{(r-1)}(x)$ exists at each point of a set E of positive measure. Then necessary and sufficient for the rth conjugate $\tilde f_r(x)$ to exist almost everywhere in E is that the indefinite integral of f has a generalized derivative of order $r+1$ almost everywhere in E. The proof of sufficiency is comparatively simple and is given by real-variable methods. The proof of necessity is carried out by complex methods and makes use of Cesàro summability of the conjugate series. A similar argument and application of the principal theorem yield the following theorem. Suppose that $f_{(j)}(x)$ ($j=1, 2, \cdots, r-1$) exist in a set E and that at each $x\in E$ the integral $-\pi^{-1}\int_\varepsilon^\infty \delta_r(x, t)t^{-1}dt$ remains bounded as $\varepsilon\to +0$. Then $\tilde f_r(x)$ exists almost everywhere in E.

J. G. Herriot (Stanford, Calif.)

3935:

Lorch, Lee. The Lebesgue constants for Jacobi series. II. Amer. J. Math. **81** (1959), 875–888.

[For part I see Proc. Amer. Math. Soc. **10** (1959), 756–761; MR **22** #864.]

In the present paper the Lebesgue constants of the Jacobi series, i.e., the expressions

$$L_n(\alpha, \beta) = \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)} \times \int_0^\pi (\sin \tfrac12\theta)^{2\alpha+1}(\cos \tfrac12\theta)^{2\beta+1}|P_n^{(\alpha+1,\beta)}(\cos\theta)|d\theta,$$

are studied on the range $-\frac12<\alpha<\frac12$, $\alpha-\beta<1$, $\rho>-1$. It is shown that

$$L_n(\alpha, \beta) = A(\alpha, \beta)n^{\alpha+1/2}+B(\alpha)+O(n^{-1/2})+O(n^{\alpha-\beta-1}).$$

Here $A(\alpha, \beta)$ is an elementary constant expressible in terms of Γ-functions and $B(\alpha)$ can be written as an infinite series whose terms involve certain Bessel functions and their zeros.

G. Szegő (Stanford, Calif.)

3936:

Stečkin, S. B. On Fourier coefficients of continuous functions. Amer. Math. Soc. Transl. (2) **14** (1960), 311–332.

Translation of Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 93–116 [MR **19**, 31].

3937:

Telyakovskiĭ, S. A. Approximations to differentiable functions by linear means of their Fourier series. Izv. Akad. Nauk SSSR. Ser. Mat. **24** (1960), 213–242. (Russian)

Let $W^{(r)}$ denote the class of 2π-periodic functions f with $f^{(r-1)}$ absolutely continuous and $|f^{(r)}(x)|\leqq 1$ a.e. on $(0, 2\pi)$. Let $\Lambda=(\lambda_{n,k})$ be a triangular matrix with $\lambda_{n,1}=1$ for all n. The Λ means u_n of the Fourier series for a function f are defined as

$$u_n(\Lambda, f, x) = 2^{-1}\lambda_{n,1}a_0+\sum_{k=1}^{n-1}\lambda_{n,k+1}(a_k\cos kx+b_k\sin kx).$$

Let $U_n(\Lambda)=\sup_{f\in W^{(r)}}\|f(x)-u_n(\Lambda, f, x)\|$, where $\|\cdot\|$ denotes the sup norm. By involved integral computations the author obtains asymptotic expressions for $U_n(\Lambda)$. When Λ is appropriately specialized, the u_n become $v_{n,m}(f, x)$—the de La Vallée-Poussin means—defined by $v_{n,m}(f, x)=nm^{-1}\sigma_n(f, x)-m^{-1}(n-m)\sigma_{n-m}(f, x)$, where σ_n is the nth $(C, 1)$ mean. Precise asymptotic estimates are obtained for $V_{n,m}=\sup_{f\in W^{(r)}}\|f(x)-v_{n,m}(f, x)\|$ under the assumption that $\lim mn^{-1}$ exists and is equal to θ, $0\leqq\theta\leqq 1$. For example, if $\theta=0$ then

$$V_{n,m} = 4\pi^{-2}n^{-r}\ln(nm^{-1})+O(n^{-r}).$$

R. R. Goldberg (Evanston, Ill.)

3938:

Mattila, P. The relation between the real and imaginary parts of the transfer function. Arkhimedes **1960**, no. 1, 36–44. (Finnish)

The relation between the real and imaginary parts of the transfer function is established in a relatively simple manner, with the help of appropriately chosen Fourier transforms. *O. Lehto* (Helsinki)

3939:

Levin, B. Ya. On a uniqueness theorem in harmonic analysis. Vestnik Leningrad. Univ. **14** (1959), no. 13, 59–62. (Russian. English summary)

It is proved that if the function $\varphi(z)=\int_{-\infty}^{\infty} e^{itz}f(t)dt$ is bounded on the imaginary axis and if $f(t)$ is continuous with $|f(t)|=O(\exp(-kt^2))$ for any $k>0$, then $f(t)\equiv 0$. The result is best possible in the sense that, for any λ $(1<\lambda<2)$, there exists a continuous $f(t)$, not identically 0, such that $|f(t)|=O(\exp(-|t|^\lambda))$ and $\varphi(iy)$ is bounded over $(-\infty<y<\infty)$. The proof is based on the lemma that if $\varphi(z)$ is an entire function with $\int_{-\infty}^{\infty} |\varphi(x+iy)|^2dx< c\exp(2\sigma|y|^q)$, then

$$\left|(2h)^{-1}\int_{t-h}^{t+h} f(t)dt\right| < c_1\exp(-(\sigma_1-\varepsilon)|t|^p),$$

where $1/p+1/q=1$ and $\sigma_1=1/p(q\sigma)^{p-1}$.

A. J. Lohwater (Houston, Tex.)

3940:

Hsiang, Fu Cheng. An inequality for Fourier transforms. Portugal. Math. **18** (1959), 87–89.

Let $f(x)=\int e^{ixt}\varphi(t)dt$, with $\varphi\in L_1$ and f of compact support. The author's inequality would imply that $|f(0)|\leqq(\pi/4)\int|\varphi|$, and is therefore incorrect. (For $\varphi(t)=\sin^2 t/t^2$ one has $f(0)=\int|\varphi|$.)

J. Korevaar (Madison, Wis.)

3941:

Tornehave, Hans. On entire functions almost periodic in two directions. Math. Scand. **6** (1958), 160–174.

The class of entire functions $f(z)=f(x+iy)$ almost periodic in every strip $|x|<a$ and in every strip $|y|<b$ is denoted by $\mathscr{D}$. That $\mathscr{D}$ contains non-constant functions was first proved by R. Petersen [Mat.-Fys. Medd. Danske Vid. Selsk. **15** (1938), no. 8, 1–25]. The functions $f(z)$ of $\mathscr{D}$ have two Fourier-Dirichlet series, $\sum A_n\exp(\Lambda_n z)$ and $\sum A_n'\exp(i\Lambda_n' z)$ corresponding to the two directions of almost periodicity; and $f(z)$ is said to have the integral basis $\alpha_1, \alpha_2, \cdots$ in the strip $|x|<a$ if the α's are rationally independent and if each Λ_n can be expressed in the form $\Lambda_n=r_{n1}\alpha_1+\cdots+r_{nk_n}\alpha_{k_n}$, where all the r's are integers. An integral basis in the strip $|y|<b$ is similarly defined. The object of this paper is to prove the existence of non-constant functions of $\mathscr{D}$ with integral bases in one or in both directions. The author remarks that, if a non-constant function of $\mathscr{D}$ has an integral basis, then this basis consists of at least two members. The functions here constructed have, in fact, two-term bases. An indication is given of how the method used in the construction can be adapted to give functions with various other properties. Some still open questions in this field are also mentioned. *H. Burkill* (Sheffield)

3942:

Berman, D. L. Linear trigonometric polynomial operations in spaces of almost-periodic functions. Mat. Sb. (N.S.) **49 (91)** (1959), 267–280. (Russian)

Soit $L_n(f,x)$ le polynôme d'interpolation de Lagrange d'ordre n. L'auteur généralise la formule de Faber-Marcinkiewicz

$$(2\pi)^{-1}\int_0^{2\pi} L_n(f_t, x-t)dt = S_n(f,x),$$

où $f(x)$ est une fonction périodique (mod 2π), $S_n(f,x)$ la somme partielle de sa série de Fourier, $f_t=f(x+t)$. Sa généralisation donne la formule correspondante pour des fonctions presque-périodiques (f.p.p.). Soient E l'ensemble des f.p.p. (par exemple au sens de V. Stepanoff); $\{\lambda_k\}_1^n$, ensemble de nombres réels, le spectre du polynôme trigonométrique généralisé (p.t.g.) $P(t)=\sum_1^n a_k\exp\{\lambda_k xi\}$; et $C(P)$ l'ensemble correspondant des fonctions $\{\exp(\lambda_k xi)\}_1^n$. Soit $M\{f(x)\}=\lim\{T^{-1}\int_0^T f(x)dx\}$, $T\to\infty$, la moyenne d'une f.p.p. et $K(x)$ un p.t.g. fixe.

L'auteur introduit d'abord la notion d'opération polynômiale trigonométrique $U(f,x)$ du type K. Elle est linéaire avec le contredomaine E. Pour tout $f\in E$, $U(f,x)$ est un p.t.g. avec $C(P)\subset C(K)$. Pour tout p.t.g. $T(x)$ avec $C(T)\subset C(K)$ on a $U(T,x)=M_t\{T(x-t)K(t)\}$. Si $f\in E$, la moyenne de f soit définie par

$$\mathfrak{S}(f)=\mathfrak{S}(f,x)=M_t\{f(x-t)K(t)\},$$

et elle est aussi un p.t.g. On a alors pour $f\in E$

$$M_t\{U(f_t,x-t)\}=M_t\{f(x-t)K(t)\}=\mathfrak{S}(f,x).$$

En prenant pour la norme d'une f.p.p. au sens de Stepanoff, $p\geqq 1$, $\|f\|_{S_L^p}=\sup_x[L^{-1}\int_x^{x+L}|f(\zeta)|^pd\zeta]^{1/p}$, on a aussi $\|U\|\geqq\|\mathfrak{S}\|$.

L'auteur montre ensuite que ces résultats ont lieu aussi pour des f.p.p. définies sur les groupes abéliens. La moyenne d'une telle fonction est définie à l'aide de celle de von Neumann. *M. Tomić* (Belgrade)

3943:

Kultze, Rolf. Fastperiodische Kompaktifikation von Halbgruppen. Math. Ann. **139**, 44–50 (1959).

Principal results: Let H be a semigroup with identity element, and let B be an algebra of complex-valued Maak almost periodic functions on H such that $1\in B$ and B is closed under complex conjugation, uniform convergence, and bilateral translations; then there exists a compact topological group H^* and a homomorphism σ of H onto a dense subsemigroup of H^* such that a complex-valued function φ on H belongs to B if and only if $\varphi=\tilde\varphi\sigma$ for some complex valued continuous function $\tilde\varphi$ on H^*.

An additional result: A compact semigroup H with identity element is a topological group if and only if every complex-valued continuous function on H is Maak almost periodic. *W. H. Gottschalk* (New Haven, Conn.)

3944:

Buchwalter, Henri. Saturation sur un groupe abélien localement compact. C. R. Acad. Sci. Paris **250** (1960), 808–810.

Let G be a locally compact Abelian group with dual group $\hat G$; $\{M_t\}_{t\in T}$ a family of measures on G $(T\subset R,$

sup $T=\infty$). Let E denote $L^p(G)$ $(1\leq p\leq 2)$. If $E=L^1$, let $F=C_0(G)$, the space of complex continuous functions vanishing at ∞. If $E=L^p$ $(1<p\leq 2)$, let $F=L^q$, $q=p/(p-1)$. Let $\hat{\nu}$ and $\hat{f}$ denote the Fourier-Stieltjes transform and Fourier transform of a measure ν on G and a function f on G, respectively.

Suppose that there is a function $\rho(t)>0$, $\rho=o(1)$ $(t\to\infty)$ and a function $\lambda(\hat{x})$ on $\hat{G}$ such that $\lim_{t\to\infty} 1-\hat{\mu}_t(\hat{x})/\rho(t)=\lambda(\hat{x})\neq 0$ for all $\hat{x}\neq\hat{e}$, $\hat{e}$ the identity in $\hat{G}$. Theorem 1: If $\|f-f\star\mu_t\|_E=o(\rho(t))$, then $f=0$ in E. Theorem 2: If $\|f-f\star\mu_t\|_E=O(\rho(t))$, then there is a g in the conjugate space F' of F such that $\lambda\hat{f}=\hat{g}$. Suppose also that $(1-\hat{\mu}_t)/\lambda\rho(t)=\hat{\nu}_t$, where each ν_t is a measure on G and $\|\nu_t\|$ is bounded. Then (theorem 3): If $f\in E$ and there is a $g\in F'$ such that $\lambda\hat{f}=\hat{g}$, then $\|f-f\star\mu_t\|_E=O(\rho(t))$.

Other assertions are made about the case in which E is the space of measures on G, but they are not clear to the reviewer. Some proofs are sketched.

E. Hewitt (Seattle, Wash.)

3945:

Reiter, H. Beiträge zur harmonischen Analyse. V. Math. Ann. **140** (1960), 422–441.

[For part IV see same Ann. **135** (1958), 467–476; MR **21** #2868.]

Let G be a locally compact abelian group, with dual group $\hat{G}$. Let Γ be a closed subgroup of $\hat{G}$, let $\hat{H}=\hat{G}/\Gamma$, and let H be its dual group. Let Λ be a closed, countable, affine-independent subset of $\hat{H}$, and let Ω be the complete inverse image of Λ in $\hat{G}$. (In an additive abelian group, a set is 'independent' if every non-trivial finite linear combination of its elements, with integer coefficients, is non-zero, and is 'affine-independent' if it becomes independent upon subtraction of a fixed element of the set from every other element of the set.) Let I be the ideal in $L^1(G)$ consisting of all f whose fourier transforms vanish identically on Ω. The author's theorem states that the quotient algebra $L^1(G)/I$ is isomorphic and isometric to the convolution algebra of all integrable $C_0(\Lambda)$-valued functions on the group H. In other words, it can be thought of as a tensor product $L^1(H)\otimes C_0(\Lambda)$. Two previously published results of the author about quotient algebras of $L^1(G)$ follow by specializing Λ or Γ to be $\{0\}$.

H. Mirkil (Hanover, N.H.)

3946:

Willcox, Alfred B. Silov type C algebras over a connected locally compact abelian group. Pacific J. Math. **9** (1959), 1279–1294.

Let G be a connected, locally compact, Abelian group, $\{C_n\}$ a sequence of increasing compact neighborhoods of the identity whose union is G. Let K be a primary commutative Banach algebra with maximal ideal Q, an identity e, and norm $|x|$. Let ω be a homomorphism of the character group $\hat{G}$ of G into the coset $e+Q$ in K. For a function f on G, let $[f]^{(n)}(t)=f(t)$ for $t\in C_n$ and 0 otherwise. Consider an arbitrary sequence of complex linear combinations of continuous characters χ of G: $\{\sum_{j=1}^{k_n} c_{jn}\chi_{jn}\}_{n=1}^{\infty}$. Let $f^{(n)}=[\sum c_{jn}\chi_{jn}]^{(n)}$. Let

$$N(f^{(n)}-f^{(m)}) = \sup_{t\in G}\left|\sum c_{jn}[\chi_{jn}]^{(n)}(t)\omega(\chi_{jn})-\sum c_{jm}[\chi_{jm}]^{(m)}(t)\omega(\chi_{jm})\right|.$$

The number $N(f^{(n)})$ is defined analogously. The sequence $\{f^{(n)}\}$ of functions on G is called ω-Cauchy if $N(f^{(n)}-f^{(m)})\to 0$, and $\|\{f^{(n)}\}\|$ is defined as $\lim N(f^{(n)})$. Defining $(f-g)^{(n)}$ in the obvious way, where $g^{(n)}$ is constructed from another sequence of linear combinations of characters, one defines $\{f^{(n)}\}$ and $\{g^{(n)}\}$ as being equivalent if both are ω-Cauchy and $\|\{(f-g)^{(n)}\}\|=0$. The set of equivalence classes so obtained is denoted by $K_\omega(G)$. With the natural operations, $K_\omega(G)$ is a commutative complex Banach algebra, normed by $\|\{f^{(n)}\}\|$. It is independent of the sequence $\{C_n\}$. This construction generalizes a construction of G. E. Šilov [Uspehi Mat. Nauk **6** (1951), no. 1 (41), 91–137; MR **13**, 139].

The paper is devoted to a study of $K_\omega(G)$ and certain subalgebras of $K_\omega(G)$. For example, if ω is continuous and $K_\omega(G)$ is regular and semisimple, then it is an algebra of type C. Many other facts about and characterizations of $K_\omega(G)$ are also obtained.

E. Hewitt (Seattle, Wash.)

INTEGRAL TRANSFORMS AND OPERATIONAL CALCULUS

3947a:

Kreĭn, M. G. The ideas of P. L. Čebyšev and A. A. Markov in the theory of limiting values of integrals and their further development. Amer. Math. Soc. Transl. (2) **12** (1959), 1–121.

3947b:

Kreĭn, M. G.; Rehtman, P. G. Development in a new direction of the Čebyšev-Markov theory of limiting values of integrals. Amer. Math. Soc. Transl. (2) **12** (1959), 123–135.

The Russian originals [Uspehi Mat. Nauk (N.S.) **6** (1951), no. 4 (44), 3–120; **10** (1955), no. 1 (63), 67–78] have already been reviewed [MR **13**, 445; **16**, 1005].

3948:

Muckenhoupt, Benjamin. On certain singular integrals. Pacific J. Math. **10** (1960), 239–261.

The one-dimensional transform $\tilde{f}(x)$ is defined by $\tilde{f}(x)=\int_0^\infty f(x-t)/t^{1+i\gamma}\,dt$. The integral is given a meaning by some method of summation, such as logarithmic Abel summation defined by $(S)\int_0^1 g(t)\,dt=\lim_{\epsilon\to 0}\int_0^1 t^\epsilon g(t)\,dt$.

The n-dimensional transform $\tilde{f}(x)$ is defined by

$$\tilde{f}(x) = \int_{E^n}\frac{f(x-t)\Omega(t)}{|t|^{n+i\gamma}}\,dt,$$

where E^n denotes Euclidean n-space, $x=(x_1, x_2, \cdots, x_n)$, $|x|=(x_1^2+\cdots+x_n^2)^{1/2}$, $dx=dx_1dx_2\cdots dx_n$ and $\Omega(t)=\Omega(t/|t|)$ is a function of angle which is integrable over the unit sphere. The part of the integral for which $0\leq|t|\leq 1$ can be obtained by the same summation methods as mentioned above for the one-dimensional case.

The author develops properties of these transforms in a sequence of eight theorems which mainly deal with inequalities and the existence of certain limits. Theorem 8 is as follows: Let $f(x)$ belong to L^p $(1<p<\infty)$ in E^n. Let $\Omega(t)=\Omega(t/|t|)$ be merely integrable on the unit sphere Σ. Then if

$$\tilde{f}_\epsilon(x) = \int_{|t|>\epsilon}\frac{\Omega(t)}{|t|^{n+i\gamma}}f(x-t)\,dt-\frac{f(x)}{i\gamma\epsilon^{i\gamma}}\int_\Sigma\Omega(t)\,d\sigma,$$

it satisfies

$$\|\tilde{f}_\varepsilon(x)\|_p \le \frac{C(|\gamma|+1)^2p^2}{|\gamma|(1-p)}\|f(x)\|_p,$$

where C depends only upon Ω. As $\varepsilon\to0$, $\tilde{f}_\varepsilon(x)$ converges in L^p norm to a function $\tilde{f}(x)$. Furthermore, $\|\sup_\varepsilon|\tilde{f}_\varepsilon(x)|\|_p\le c\|f(x)\|_p$, where c is independent of f, and $\tilde{f}_\varepsilon(x)$ converges almost everywhere to $\tilde{f}(x)$ as $\varepsilon\to0$.

The author then defines transforms which he calls transforms of fractional integral type. For these transforms he proves three theorems somewhat analogous to the theorems proved for $\tilde{f}(x)$ defined above.

C. Fox (Montreal)

3949:

Mairhuber, J. C.; Schoenberg, I. J.; Williamson, R. E. On variation diminishing transformations of the circle. Rend. Circ. Mat. Palermo (2) 8 (1959), 241–270.

Let $f(t)$ be a continuous function with period 2π defined for $-\infty<t<\infty$, and let $N(f)$ be the number of changes of sign of $f(t)$ in any interval of length 2π; $N(f)$ has one of the values $0, 2, 4, \cdots, +\infty$. Let $\Omega(t)$ be a bounded function with period 2π defined for $-\infty<t<\infty$, and such that at each point $\Omega(t)=\frac{1}{2}[\Omega(t+)+\Omega(t-)]$. $\Omega(t)$ is said to be variation diminishing, $\Omega\in$ V.D., if $N(g)\le N(f)$ whenever $g(t)=\int_{-\pi}^{\pi}\Omega(t-u)f(u)du$. The fundamental problem treated in the present paper is that of finding necessary and sufficient conditions for $\Omega\in$ V.D. There are two types of conditions—conditions bearing directly on $\Omega(t)$, and conditions bearing on its Fourier coefficients $\Omega\hat{}(n)$. One of the principal results of this paper is the following. Let k be a non-negative integer and let (*) $t_1<t_2<\cdots<t_{2k+1}<t_1+2\pi$, $\tau_1<\tau_2<\cdots<\tau_{2k+1}<\tau_1+2\pi$. We set $D_{2k+1}(t,\tau)=\det[\Omega(t_i-\tau_j)]$. $\Omega(t)$ is said to be totally positive if for every k and every set (*) $D_{2k+1}(t,\tau)\ge0$. The authors prove that $\Omega\in$ V.D. if and only if Ω or $-\Omega$ is totally positive. The largest value of k such that $D_{2k+1}(t,\tau)$ is not identically zero is called the order m of Ω. If Ω is of order m then it is a trigonometrical polynomial of degree m. The following conditions on $\Omega\hat{}(n)$ are sufficient for $\Omega\in$ V.D. (i) If $\Omega\hat{}(n)=\Psi(in)^{-1}$ where $\Psi(s)=e^{-cs^2+ds}\prod_1^\infty(1+d_js)e^{-d_js}$ and $c\ge0$, d_j real, and $\sum|d_j|<\infty$, then $\Omega\in$ V.D. This is an old result due to Pólya. (ii) Given any set of positive numbers $\varepsilon_1>\varepsilon_2>\cdots$, $\sum_1^\infty\varepsilon_j<\infty$, it is possible to construct a sequence r_n with $0<r_n\le\varepsilon_n$ such that if $\Omega\hat{}(0)=1$, $\Omega\hat{}(n)=\overline{\Omega\hat{}(-n)}$, $|\Omega\hat{}(n)|=r_n$ then $\Omega\in$ V.D. The authors also obtain the following necessary condition: if $\Omega\in$ V.D. then $|\Omega\hat{}(n)|\ge|\Omega\hat{}(n+1)|$ $(n=0,1,\cdots)$. (Note that trivially $\Omega\hat{}(n)=\overline{\Omega\hat{}(-n)}$.) Thus despite some progress the problem of obtaining necessary and sufficient conditions for $\Omega\in$ V.D. in terms of $\Omega\hat{}(n)$ is still open. The analogous problems which arise when the torus group is replaced by the additive group of the real numbers or by the additive group of the integers have however been solved completely by Schoenberg and Edrei respectively. [See I. J. Schoenberg, J. Analyse Math. **1** (1951), 331–374; MR **13**, 923; A. Edrei, Trans. Amer. Math. Soc. **74** (1953), 367–383; MR **14**, 853.]

I. I. Hirschman, Jr. (Huntington, W.Va.)

3950:

Džrbašyan, M. M. Integral transforms with Volterra kernels. Izv. Akad. Nauk SSSR. Ser. Mat. **24** (1960), 387–420. (Russian)

The paper is summarized in Dokl. Akad. Nauk SSSR **124** (1959), 22–25 [MR **21** #273].

R. P. Boas, Jr. (Evanston, Ill.)

3951:

Li, Ta. A new class of integral transforms. Proc. Amer. Math. Soc. **11** (1960), 290–298.

Solutions are obtained for certain singular integral equations, belonging to a class that is of importance in aerodynamical applications.

Let $c>0$ and let I be the interval $(c, 1)$. Let $f_n(\sigma)$ be defined on I, let $(d/d\sigma)[\sigma^nf_n(\sigma)]$ be piecewise continuous on I, and let $f_n(1)=0$.

It is proved that if T_n is the Čebyšev polynomial of the first kind and nth degree, and if

$$\int_\sigma^1\frac{T_n(u/\sigma)y_n(u)du}{(u^2-\sigma^2)^{1/2}}=f_n(\sigma)$$

for σ in I, then

$$y_n(u)=-\frac{2}{\pi}\int_u^1\frac{T_{n-1}(u/v)d[v^nf_n(v)]}{v^{n-1}(v^2-u^2)^{1/2}}.$$

F. Goodspeed (Quebec)

3952:

Arya, Suresh Chandra. Abelian theorem for a generalization of Laplace transformation. Collect. Math. **11** (1959), 3–12.

In this paper, the author studies the integral

$$f(s)=\int_0^\infty e^{-st}(st)^{2m}\Psi(\tfrac{1}{2}-k+m,\,2m+1;\,st)d\alpha(t),$$

where Ψ denotes Tricomi's confluent hypergeometric function given by

$$W_{k,m}(z)=e^{-z/2}z^{m+1/2}\Psi(\tfrac{1}{2}-k+m,\,2m+1;\,z).$$

He proves that, under certain conditions,

$$\limsup_{s\to0;\,s\to\infty}|s^{\nu-k-m+1/2}f(s)-A|\le$$

$$\limsup_{t\to\infty;\,t\to0}\left|\frac{\alpha(t)\Gamma(\nu-k-m+\frac{1}{2})\Gamma(\nu-k+\frac{1}{2}+m)}{\Gamma(\nu-2k+1)t^{\nu-k-m+1/2}}-A\right|.$$

B. Mohan (Benares)

3953:

Gandin, L. S.; Solovelčik, R. È. Multidimensional symmetric δ-functions. Zap. Leningrad. Gorn. Inst. **36** (1958), no. 3, 13–15. (Russian)

INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

See also 3892.

3954:

Gohberg, I. C.; Kreĭn, M. G. Systems of integral equations on a half line with kernels depending on the difference of arguments. Amer. Math. Soc. Transl. (2) **14** (1960), 217–287.

Translation of Uspehi Mat. Nauk **13** (1958), no. 2 (80), 3–72 [MR **21** #1506].

3955:

Gel'fond, A. O. On estimation of certain determinants and the application of these estimations to the distribution of eigenvalues. Amer. Math. Soc. Transl. (2) **12** (1959), 163–179.

The Russian original [Mat. Sb. (N.S.) **39** (81) (1956), 3–22] was reviewed as MR **21** #3739.

3956a:

Stesin, I. M. Computation of eigenvalues by means of continued fractions. Amer. Math. Soc. Transl. (2) **12** (1959), 141–148.

3956b:

Stesin, I. M. An estimate of the precision of computation of eigenvalues by means of continued fractions. Amer. Math. Soc. Transl. (2) **12** (1959), 149–154.

The Russian originals [Uspehi Mat. Nauk (N.S.) **9** (1954), no. 2 (60), 191–198; Vyčisl. Mat. Vyčisl. Tehn. **2** (1955), 145–150] have already been reviewed [MR **16**, 405; **17**, 414].

3957:

Grebenyuk, D. G. On approximate solution of Fredholm integral equations. Izv. Akad. Nauk UzSSR. Ser. Fiz.-Mat. **1959**, no. 1, 63–68. (Russian. Uzbek summary)

The author applies the method of uniform approximations by L. G. Šnirel'man [see Izv. Akad. Nauk SSSR. Ser. Mat. **2** (1938), 53–59] to the solution of Fredholm nonhomogeneous integral equations.

S. Kulik (Long Beach, Calif.)

3958:

Baluev, A. N. Application of semi-ordered norms in approximate solution of non-linear equations. Leningrad. Gos. Univ. Uč. Zap. Ser. Mat. Nauk **33** (1958), 18–27. (Russian)

Approximation methods of L. V. Kantorovič [Dokl. Akad. Nauk SSSR **76** (1951), 17–20; **80** (1951), 849–852; MR **12**, 835; **13**, 469] are applied to the equation $x(s) = \int_0^1 K(s, t, x(t))dt + f(s)$. Several examples are worked out.

E. Hewitt (Seattle, Wash.)

3959:

Baluev, A. N. Approximate solution of non-linear integral equations. Leningrad. Gos. Univ. Uč. Zap. Ser. Mat. Nauk **33** (1958), 28–31. (Russian)

Consider the (in general) nonlinear integral equation

$$(1) \qquad x(s) = \int_0^1 K(s, t, x(t))dt + f(s), \quad 0 \le s \le 1.$$

Consider also the equations in the unknowns $x_1, \cdots, x_n$

$$(2) \qquad x_i - \sum_{k=1}^{n} A_k K(t_i, t_k, x_k) - f(t_i) = 0,$$

where $0 \le t_1 < t_2 < \cdots < t_n \le 1$, and the A_k's are coefficients of a certain quadrature formula. Conditions are given under which (2) admits a solution "near" to a solution of (1).

E. Hewitt (Seattle, Wash.)

3960:

Nehari, Zeev. On a class of nonlinear integral equations. Math. Z. **72** (1959/60), 175–183.

The author studies the non-linear integral equation

$$(1) \qquad y(x) = \int_a^b K(x, t)y(t)F[y^2(t), t]dt,$$

where the kernel $K(x, t)$ is symmetric, positive definite and continuous, while $F(u, x)$ is positive and continuous for $a \le x \le b$ and for all real non-negative u, and satisfies the condition (2) $u_1^{-\varepsilon}F(u_1, x) < u_2^{-\varepsilon}F(u_2, x)$ when $0 < u_1 < u_2 < \infty$, where ε is a fixed positive number. With these assumptions it is proved that (1) has a continuous, nontrivial solution. The proof is based on the variational problem of minimizing the functional

$$H(y) = \int_a^b [y^2F(y^2, x) - G(y^2, x)]dx,$$

subject to the side conditions $\int_a^b y^2F(y^2, x)dx = J(y, y)$, $y(x) \not\equiv 0$, where $G(u, x) = \int_0^u F(s, x)ds$ and $J(u, v) = \int_a^b\int_a^b K(x, t)u(x)F(u^2, x)v(t)F(v^2, t)dxdt$. This theorem is then extended to the case in which $yF(y^2, x)$ contains a linear term, under suitable restrictions. The paper is self-contained.

D. H. Hyers (Berkeley, Calif.)

3961:

Valic'kii, Yu. M. Functions analytic with respect to some integro-differential operators and their applications. Dopovidi Akad. Nauk Ukrain. RSR **1959**, 237–240. (Ukrainian. Russian and English summaries)

Author's summary: "An operator (1) $\Lambda(y) = y^{(n)}(x) + \sum_{i=0}^{n-1} p_i(x)y^{(i)}(x) + \sum_{i=0}^{n-1} \int_{x_0}^x H_i(x, t)y^{(i)}(t)dt$ is considered, the functions $p_0(x), \cdots, p_{n-1}(x)$ being continuous and $H_0(x, t), \cdots, H_{n-1}(x, t)$ measurable and bounded for $a < x, t < b$; $x_0 \in (a, b)$. A number of facts which hold in the theory of ordinary analytical functions are proved for functions which can be expanded into the series $f(x) = \sum_{m=0}^{\infty} a_m\varphi_m(x, x_0)$ where $\{\varphi_m(x, x_0)\}$ is a Λ-basis [M. K. Fage, Trudy Moskov. Mat. Obšč. **7** (1958), 227–268; MR **21** #2091]. The local equivalence of any two operators of type (1) is established.

"The local equivalence of an operator

$$Sf(x) = \int_{x_0}^x S(x, t)f(t)dt$$

under two conditions with an operator of n-fold integration is proved as an application."

FUNCTIONAL ANALYSIS

See also 3813, 3961.

3962:

Gordon, Hugh. Topologies and projections on Riesz spaces. Trans. Amer. Math. Soc. **94** (1960), 529–551.

Let E be a Riesz space. A topology T on E is said to be adequate if: (A_1) T is compatible with the vector space structure of E; (A_2) the mappings $(x, y) \to \sup(x, y)$ and $(x, y) \to \inf(x, y)$ of $E \times E$ into E are continuous; (A_3) there is a fundamental system B of the origin such that $U \in B$, $y \in U$, $x \in E$ and $0 \le x \le y$ imply $x \in U$. If E is a Riesz

space and T an adequate topology on E then E_T' is the vector space of all T-continuous linear forms on E. This paper is essentially centered around the following problem. (P) Let E be a Riesz space and T an adequate topology on E. Let S be a second adequate topology on E and M the vector space of all $x' \in E_T'$ which are S-continuous. Determine under what conditions there is a projection P of E_T' onto M such that $P \geqq 0$, $I-P \geqq 0$.

Here are some of the main results of the paper, which in particular give a complete solution to the problem (P). (I) Let E, T, S and M be as in the statement of the problem (P). Then there is a projection P of E_T' onto M, with $P \geqq 0$, $I-P \geqq 0$, if and only if (T, S) satisfies the condition: If $y' \in E_T'$, $y' \geqq 0$, and if $A=\{x|y'(x) \leqq 1\}$ belongs to every filter on E, bounded from above, which converges to 0 for the topology defined on E by the set of semi-norms $\{x \to x'(|x|)|x' \in M, x' \geqq 0\}$, then $y' \in E_S'$. (II) Let E be a Riesz space and D a set of directed families of positive elements of E satisfying the conditions: (1) $(x/n)_{n=1,2,\cdots} \in D$ for every $x \in E$; (2) if $(x_j) \in D$ and $a \geqq 0$ then $(ax_j) \in D$; (3) if $(x_j) \in D$ and (y_j) is such that $0 \leqq y_j \leqq x_j$ for all j then $(y_j) \in D$. Under this hypothesis there is on E an adequate topology T such that $x' \in E_T'$ if and only if $(x'(x_j))$ converges to 0 for every $(x_j) \in D$. (III) Let E be a Riesz space, T an adequate topology on E, D a set of bounded directed families of positive elements of E having the properties (1), (2), (3) stated above, and M the vector space of all $x' \in E_T'$ such that $(x'(x_j))$ converges to 0 for every $(x_j) \in D$. Then there is a projection P of E_T' onto M with $P \geqq 0$, $I-P \geqq 0$. (IV) Let E be a Riesz space and T an adequate topology on E. Let P be a projection of E_T' into E_T', such that $P \geqq 0$, $I-P \geqq 0$. Then there is a set D of bounded directed families of elements of E such that x' belongs to the range of P if and only if $(x'(x_j))$ converges to 0 whenever $(x_j) \in D$.

C. Ionescu Tulcea (New Haven, Conn.)

3963:

Gordon, Hugh. Measures defined by abstract L_p spaces. Pacific J. Math. **10** (1960), 557–562.

Let L be a Riesz space (i.e., a linear lattice) whose elements are functions f on a set E (the operations are the usual pointwise operations), let $p \geqq 1$ be fixed, and let $\|f\|$ be a semi-norm on L such that (1) L is complete with respect to this semi-norm, (2) f and $|f|$ have the same semi-norm, (3) $\|f+g\|^p \geqq \|f\|^p + \|g\|^p$ for f, g non-negative, (4) $\|f+g\|^p \leqq \|f\|^p + \|g\|^p$ for f, g non-negative and $f \wedge g = 0$, (5) $1 \in L$ and $\|1\|=1$. The main theorem is that in this case there is a countably additive measure μ in E such that L is essentially (i.e., up to μ-null functions) the space $L_p(\mu)$. It is also proved that if instead of the semi-norm there is defined on L a non-negative definite bilinear form such that (i) L is complete with respect to the semi-norm $\|f\|$ determined by the form, (ii) $0 \leqq f \leqq g$ implies $\|f\| \leqq \|g\|$, (iii) $\|f \vee 0\| \leqq \|f\|$, (iv) $1 \in L$ and $\|1\|=1$, then L is essentially the space $L_2(\mu)$. The extension to abstract Riesz spaces is indicated, so that an extension of S. Kakutani's result for $p=1$ [Ann. of Math. (2) **42** (1941), 994–1024; MR **3**, 205] is obtained.

A. C. Zaanen (Pasadena, Calif.)

3964:

Phelps, R. R. Uniqueness of Hahn-Banach extensions and unique best approximation. Trans. Amer. Math. Soc. **95** (1960), 238–255.

Let E be a Banach space. We say that a subspace $M \subset E$ has property U if each linear functional on M has a unique norm-preserving extension to E. M has property H if for each $x \in E$ there exists a unique $y \in M$ such that $|x-y| = d(x, M)$. Now M has property U in E if and only if its annihilator $M^\perp$ has property H in E'. A partial converse: if M is non-H, then $M^\perp$ is non-U. There exists a pair $M \subset E$ such that M is H while $M^\perp$ is non-U. If $x \in E$ let $A(x)$ be the set of those $f \in E'$ such that $|f|=1$ and $f(x)=|x|$. Similarly, for $f \in E'$ let $B(f)$ be the set of those $x \in E$ such that $|x|=1$ and $f(x)=|f|$. If M has finite deficiency n and if there exists an $f \in M^\perp$ such that $\dim B(f) \geqq n$, then M is non-H. Dually, if M has finite dimension n and if there is a non-zero $x \in M$ with $\dim A(x) \geqq n$, then M is non-U. If M is an n-dimensional subspace which is non-H, then there exist a nonzero $x \in M$ and n linearly independent extreme points f_i of the unit ball of E' such that $f_i(x)=0$ for $i=1, 2, \cdots, n$. If E is an L_1 over a non-atomic measure space, then a finite dimensional subspace M is U if and only if for each nonzero $x \in M$ the set of zeros of x is of measure zero. For non-atomic measures every subspace of L_1 with finite dimension or deficiency is non-H. If M is an n-dimensional subspace of l_1, then M is U if and only if each nonzero $x \in M$ has at most $n-1$ zeros. If M is an n-dimensional subspace of $C(T)$ with T compact Hausdorff and if M is U then each nonzero $x \in M$ has the property that $|x(t)|=|x|$ for at most n points $t \in T$. For c_0, this characterizes U. An n-dimensional subspace M of $C(T)$ has H if and only if each nonzero $x \in M$ has at most $n-1$ zeros.

V. Pták (New Orleans, La.)

3965:

Andô, Tsuyoshi. On products of Orlicz spaces. Math. Ann. **140** (1960), 174–186.

Wegen der Bezeichnungen für das Folgende s. z.B. M. A. Krasnoselkskij und J. B. Rutizkij, *Konvexe Funktionen und Orlicz-Räume* [MR **21** #5144]. Das $\odot$ Produkt $L_M^* \odot L_N^*$ der Orlicz-Räume L_M^* und L_N^* ist die Menge aller meßbaren Funktionen auf einer beschränkten, abgeschlossenen Menge G im n-dimensionalen Euklidischen Raum, welche als Linearkombination von Funktionen der Form $f(t)g(t)$ mit $f \in L_M^*$ und $g \in L_N^*$ ausgedrückt werden können. Sei $G \times G$ das topologische Produkt im $2n$-dimensionalen Euklidischen Raum, in dem das übliche Lebesgue'sche Maß definiert ist. Der Orlicz-Raum $L_M^*(G \times G)$ definiert durch $M(\xi)$ auf $G \times G$ wird mit $\hat{L}_M^*$ bezeichnet. Das $\otimes$ Produkt $L_M^* \otimes L_N^*$ der Orlicz-Räume L_M^* und L_N^* ist die Menge aller meßbaren Funktionen auf $G \times G$, welche als Linearkombination von Funktionen der Form $f(t)g(s)$ mit $f \in L_M^*$ und $g \in L_N^*$ ausgedrückt werden können. Der Verfasser untersucht Einschließungsbeziehungen zwischen $L_M^* \odot L_N^*$ und L_R^* (R ist wie N und M eine $\mathcal{N}$-Funktion) oder zwischen $L_M^* \otimes L_N^*$ und $\hat{L}_R^*$. Biespiele: (Th. 1.) $L_M^* \odot L_N^* \subset L_R^*$ genau dann, wenn $\alpha, \gamma > 0$ existieren, sodaß $R(\alpha\xi\eta) \leqq M(\xi) + N(\eta)$ für ξ, $\eta \geqq \gamma$. (Th. 6.) $L_M^* \otimes L_N^* \subset \hat{L}_R^*$ genau dann, wenn $\alpha, \gamma > 0$ existieren, sodaß $R(\alpha\xi\eta) \leqq M(\xi)N(\eta)$ für ξ, $\eta \geqq \gamma$. Einige Aussagen enthalten Verschärfungen von Aussagen des Buches von Krasnoselskij und Rutizkij.

G. Goes (Evanston, Ill.)

3966:

Andô, Tsuyoshi. Convexity and evenness in modulared semi-ordered linear spaces. J. Fac. Sci. Hokkaido Univ. Ser. I **14**, 59–95 (1959).

The author studies problems of isomorphism and duality in modulared semi-ordered spaces [as defined in Nakano, *Modulared semi-ordered linear spaces*, Maruzen, Tokyo, 1950; MR **12**, 420], with special reference to properties analogous to smoothness and rotundity in normed spaces as discussed by the reviewer in Trans. Amer. Math. Soc. **78** (1955), 516–528 [MR **16**, 716]. Two norms are attached to the modular and relations are found between rotundity or smoothness properties of the modular and of its associated norms. The results for modulared spaces seem complete; the relations of modulars with norms give answers to some questions not settled earlier in normed spaces. *M. M. Day* (Urbana, Ill.)

3967:

Granas, A. On the disconnection of Banach spaces. Fund. Math. **48** (1959/60), 189–200.

Let B be a Banach space, $X \subset B$, and call a map $f: X \to B$ a compact vector field (cvf) on X if $f(x) = x - F(x)$, where F is a compact map. The author shows essentially that cvf on subsets of B have properties analogous to those of continuous maps of subsets of E^n into the unit ball, with non-vanishing cvf analogous to maps into S^{n-1}. In fact, establishing the Borsuk extension theorem for non-vanishing cvf, and following closely the pattern of, say, Hurewicz-Wallman or Eilenberg-Steenrod for the case of E^n, he proves: (1) (Borsuk theorem) A bounded closed set $X \subset B$ separates B if and only if the space of non-vanishing cvf on X is not arc-connected. (2) (Jordan theorem) Any set homeomorphic by a cvf to the unit sphere of B disconnects B. *J. Dugundji* (Los Angeles, Calif.)

3968:

Poliščuk, E. M. On the groups which do not change a mean value of a functional. Vestnik Leningrad. Univ. **12** (1957), no. 1, 175–179, 212. (Russian. English summary)

An earlier paper of the author [Ukrain. Mat. Ž. **8** (1956), 59–75; MR **17**, 1218] gave a definition of mean value of a functional for a certain class of functionals $F = F[x(t)]$ on the set of continuous functions $x(t)$ such that $a(t) \leq x(t) \leq b(t)$, $0 \leq t \leq 1$, $b(t) - a(t) = \Gamma(t) > 0$. The explicit formula for a functional $F[x(t)]$ here was in the form of a limit of a certain expression involving a function $H(\xi_1, \cdots, \xi_n; t_1, \cdots, t_n)$ of $2n$ arguments satisfying stated conditions. It is shown that the mean value thus assigned to a functional does not change under a certain group G of permutations of the arguments of the function H entering in the expression for the functional $F[x(t)]$. A geometrical interpretation of the group G is given. *A. M. Yaglom* (RŽMat **1958** #512)

3969:

Zaidman, S. La représentation des fonctions vectorielles par des intégrales de Laplace ou de Laplace-Stieltjes. I. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **10** (1959), 173–208. (Romanian. Russian and French summaries)

This is the first part of a paper concerning the representation as Laplace or Laplace-Stieltjes transforms of certain functions on $R_+ = [0, \infty)$ to a Banach space E. Various classes of vector-valued functions with bounded variation are studied in detail (in particular certain compactness criteria are obtained). An inversion formula for vector-valued Laplace-Stieltjes transforms is given. See also a previous paper of the author in Ann. of Math. (2) **68** (1958), 260–277 [MR **21** #803]. *C. Ionescu Tulcea* (New Haven, Conn.)

3970:

de Branges, Louis. The Stone-Weierstrass theorem. Proc. Amer. Math. Soc. **10** (1959), 822–824.

The author presents another proof of the Stone-Weierstrass theorem, based on a property of extreme linear functionals. The essential idea is as follows. Let E be a subspace of $C[S]$, the space of continuous real functions on compact S. Let L be an extreme point of B, the linear functionals of norm ≤ 1 that vanish on E. Then, any bounded Borel measurable function g such that $L(fg) = 0$ for all $f \in E$, must be essentially constant with respect to L; that is, $L(fg) = L(\rho f)$ for some constant ρ, and all $f \in C[S]$. (Proof: Assume $0 \leq g \leq 1$ and set $L^*(f) = L(fg)$. With $p = \|L^*\|$, $L_1 = p^{-1}L^*$, $L_2 = (1-p)^{-1}(L - L^*)$, one has $L_i \in B$ and $L = pL_1 + (1-p)L_2$.) If now E is a separating subalgebra of $C[S]$, and B is not null, then any $g \in E$ is constant on the support of L, which is therefore a single point p, and all $f \in E$ must vanish at p. *R. C. Buck* (Madison, Wis.)

3971:

Bessaga, C.; Pełczyński, A. Spaces of continuous functions. IV. On isomorphical classification of spaces of continuous functions. Studia Math. **19** (1960), 53–62.

Let $C(Q)$ denote the Banach space of all continuous real-valued functions on a compact metric space Q. For countable Q, let $\chi(Q) = [\kappa(Q)]^\omega$, where $\kappa(Q)$ is the smallest ordinal number γ such that the γth derivative of Q is empty. The main results of the paper are: (A) Let Q and Q_1 be zero-dimensional compact metric spaces. Then $C(Q)$ and $C(Q_1)$ are isomorphic if and only if they have the same linear dimension, and if and only if one of the following conditions holds: (i) Q and Q_1 are finite and have the same number of points; (ii) Q and Q_1 are countable and $\chi(Q) = \chi(Q_1)$; (iii) Q and Q_1 are uncountable. (This gives a solution to problem 48 in the Scottish Book. The problem was proposed by Banach and Mazur.) (B) Let Q and Q_1 be arbitrary compact metric spaces. Then $C(Q)$ and $C(Q_1)$ have the same linear dimension if and only if one of the three conditions listed in (A) is satisfied.

Several unsolved problems about Banach spaces are listed at the end of the paper. *C. W. Kohls* (Rochester, N.Y.)

3972:

Kasuga, Takashi. On Sobolev-Friedrichs' generalisation of derivatives. Proc. Japan Acad. **33** (1957), 596–599.

Let G be a (not necessarily bounded) domain in R_n, A a subdomain of G, $\|f\|^A = \|f\|^{L_2(A)}$.

$$\|f\|_s^A = \sum_{0 \leq |\sigma| \leq s} \|D^\sigma f\|^A$$

(σ = multi-index), $\|\cdot\|_s^G = \|\cdot\|_s$. A function U has strong derivatives U_σ up to order s ($|\sigma| \leq s$) if $\|U_\sigma\|^A < \infty$ for all

compact $A \subset G$ and a sequence $f_m \in C^{(s)}(G)$ exists such that $\|D^\nu f_m - U_\nu\|^A \to 0$ as $m \to \infty$, for every compact $A \subset G$. (The U_ν are unique.)

The author proves that the space E_s of functions U having s strong derivatives such that $\|U\|_s < \infty$ is identical with the closure of $E_s \cap C_\infty(G)$ in the $\|\cdot\|_s$ norm. The author observes that for bounded smooth domains the result is known. *W. Littman* (Minneapolis, Minn.)

3973:

Zieleźny, Z. Über die Mengen der regulären und singulären Punkte einer Distribution. Studia Math. **19** (1960), 27–52.

A distribution T has been defined by Łojasiewicz [Studia Math. **16** (1957), 1–36; MR **19**, 433] to have a value at a point x_0 if the distributions $T(x_0+\lambda x)$, defined by $(T(x_0+\lambda x), \psi(x)) = (T, \psi[(x-x_0)/\lambda])$ have a limit as $\lambda \to 0$. The present article defines T to be bounded at x_0 if for x in a neighbourhood of 0 the set of distributions $T(x_0+\lambda x)$ is bounded for $0<\lambda<1$, and to be right [left] bounded if this holds for x to the right [left] of 0. T is bounded of order n if there is a measure μ such that $T = \mu^{(n)}$ in a neighbourhood of x_0 and $|\mu|(x_0, x) = O(|x-x_0|^{n+1})$. T has a value of order n if μ is absolutely continuous at x_0. If T is bounded both on the right and left, then $T = T_0 + \sum_1^k c_j \delta^{(j)}(x-x_0)$ where T_0 is bounded at x_0. If T is bounded of order n and has a value of order m at x_0, then $m+1 \geqq n \geqq m$.

T has a value almost everywhere on the set of points where it is at least one-sidedly bounded. A distribution bounded everywhere is determined by its values. The properties of sets of boundedness and sets of points where T has a value are discussed: typical theorems are that the set where T is bounded of order $\leqq k$ is an F_σ, the set of points where a distribution is bounded is an F_σ, the set where it is bounded but singular (i.e., without a value) is a $G_{\delta\sigma}$ and a null-set. Distributions are constructed having various properties of boundedness or regularity on given sets of particular types. *J. L. B. Cooper* (Cardiff)

3974:

Bouix, Maurice. Extension à certaines distributions de quelques formules d'analyse vectorielle. J. Math. Pures Appl. (9) **39** (1960), 63–84.

Formulae for the curl, div, grad of functions and vectors continuous and differentiable save for discontinuities on surfaces S are developed. Inter alia, if X is a vector valued, f a scalar valued function of position, with saltus $\sigma(X)$, $\sigma(f)$ on S, then $\operatorname{div} X = \{\operatorname{div} X\} + n\sigma(X)\delta_S$, $\operatorname{curl} X = \{\operatorname{curl} X\} + n \wedge \sigma(X)\delta_S$, $\Delta f = \{\Delta f\} + n\sigma(\operatorname{grad} f)\delta_S + \operatorname{div}[\sigma(f)n\delta_S]$, where quantities in $\{\,\}$ denote the ordinary derivative where it exists, and n and δ_S are respectively the normal vector and a distribution of unit mass at points of S. *J. L. B. Cooper* (Cardiff)

3975:

Shibata, Toshio. On a space of distributions defined by a weight system. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A **6**, 223–228 (1959).

A weight system is a set of positive continuous functions forming a directed system with respect to $\leqq$ (where $f \leqq g$ means that f/g is bounded), containing a cofinal sequence and satisfying other conditions. $B_w{}^0$ is the space of continuous functions majorized by w, B_w is the space of infinitely differentiable functions φ such that for any differential operator D, $D\varphi \in B_w{}^0$, and B_W is the inductive limit of the B_w. The dual space of distributions is studied. {A number of misprints make the reading of the paper rather difficult.} *J. L. B. Cooper* (Cardiff)

3976:

Krishnamurthy, V. On the spaces of certain classes of entire functions. J. Austral. Math. Soc. **1** (1959/61), 147–170.

Let $C(\rho, d)$ denote the class of all entire functions of order not greater than ρ and type not exceeding d. Let $C(\rho)$ denote the class of all entire functions of order not exceeding ρ. The reviewer has discussed complete linear metric topologies on these two classes, the metric being expressed in terms of a sequence of increasing norms on these classes [in a paper under publication in the Jubilee Volume of the J. Indian Math. Soc.]. The author considers the problem of proper bases in these two metric spaces denoted respectively by $\Gamma(\rho, d)$ and $\Gamma(\rho)$. The notion of proper bases was introduced by the reviewer for the linear metric space of all entire functions [Proc. Amer. Math. Soc. **3** (1952), 874–883; MR **14**, 657] and improved upon by Arsove [ibid. **8** (1957), 264–271; MR **19**, 259]. A sequence $[\alpha_n]$ $(n = 0, 1, 2, \cdots)$ of functions of one of these spaces is said to be linearly independent if $\sum c_n\alpha_n = 0$ implies $c_n = 0$, for $n = 0, 1 \cdots$. If in addition, $\sum c_n\alpha_n$ converges if and only if $\sum c_n e_n$ $(e_n \equiv z^n)$ converges (in the metric spaces concerned), the sequence is said to be a proper base of the subspace consisting of all convergent series of the form $\sum c_n\alpha_n$. Arsove [loc. cit] has obtained necessary and sufficient conditions for a sequence (α_n) to form a proper base in the space of all entire functions. The author obtains a similar condition for the two spaces in question. For the space $\Gamma(\rho, d)$ we quote the result. Let, for $\alpha = \sum a_n e_n$ (the usual Taylor expansion), $\|\alpha; d+\delta\| = |a_0| + \sum |a_n|[n/(d+\delta)e\rho]^{n/\rho}$. A set of necessary and sufficient conditions for (α_n) to form a proper base in $\Gamma(\rho, d)$ is (i) $\limsup_{n\to\infty} \|\alpha_n; d+\delta\|^{1/n}/n^{1/\rho} < 1/(de\rho)^{1/\rho}$ for each $\delta > 0$ and (ii) $\lim_{\delta\to 0} \liminf_{n\to\infty} \|\alpha_n; d+\delta\|^{1/n}/n^{1/\rho} \geqq 1/(de\rho)^{1/\rho}$. Similar results are proved for the space $\Gamma(\rho)$. It is proved that there is a one-to-one correspondence between the class of proper bases and homeomorphic linear mappings of these spaces into themselves. Other results discussed are conditions for a sequence of the form $\alpha_n(z) = z^n[1+\lambda_n(z)]$, $\lambda_n(0) = 0$, to form a proper base, explicit construction of a class of bases of this type, the algebraic structure introduced into the space $\Gamma(\rho)$ by natural multiplication and by Hadamard composition, and the characterisation of the automorphism group of the latter algebra as a subgroup of the group of all permutations of the set of non-negative integers.

V. Ganapathy Iyer (Annamalainagar)

3977:

Haplanov, M. G. Infinite matrices in an analytic space. Amer. Math. Soc. Transl. (2) **13** (1960), 177–183.

The Russian original was in Uspehi Mat. Nauk (N.S.) **11** (1956), no. 5 (71), 37–44 [MR **18**, 810].

3978:

Inoue, Sakuji. Normal operators in Hilbert spaces and their applications. Proc. Japan Acad. **35** (1959), 359–364.

A statement, with sketchy proofs, of some known facts about compact normal operators and integral operators.
J. Feldman (Berkeley, Calif.)

3979:

Singer, Ivan. Sur l'approximation uniforme des opérateurs linéaires compacts par des opérateurs non linéaires de rang fini. Arch. Math. **11** (1960), 289–293.

The author proves the following theorem, providing a partial answer to the unsolved problem of whether a compact linear transformation on a Banach space can always be approximated uniformly by linear transformations of finite rank. Let T be a compact linear transformation of a Banach space E into a Banach space F. Corresponding to each positive integer n and each closed linear subspace of the form $W=N+U$, where N is the null space of T and U is any finite dimensional subspace of E, there exists a transformation S of E into F with the following properties. (1) S is of finite rank, homogeneous of degree one, and uniformly continuous on E. (2) There exists a positive number M such that $\|Sx\| \leqq M\|Tx\|$ for all x in E. (3) $Sx=Tx$ for all x in W. (4) $S(x+y)=Sx+Sy$ for all x in W and y in E. (5) $\|Tx-Sx\| \leqq n^{-1}$ when $\|x\| \leqq 1$. The proof is based on the notion of a metric projection P on a finite dimensional subspace, which has been employed for a similar purpose by Aronszajn and K. T. Smith [Ann. of Math. (2) **60** (1954), 345–350; MR **16**, 488]. *D. H. Hyers* (Los Angeles, Calif.)

3980:

Lyubič, Yu. I.; Macaev, V. I. On the spectral theory of linear operators in Banach space. Dokl. Akad. Nauk SSSR **131** (1960), 21–23 (Russian); translated as Soviet Math. Dokl. **1**, 184–186.

Suppose throughout that A is a densely defined, linear operator in a Banach space $\mathfrak{B}$. Suppose further that A is a closed operator whose spectrum (denoted $\sigma(A)$) is a subset of the real line $(-\infty, \infty)$. Let $\mathfrak{B}(A)$ be the set of subspaces of $\mathfrak{B}$ that are included in the domain of A. This article concerns the set $S(A)$ of all $\mathfrak{M} \in \mathfrak{B}(A)$ such that the restriction $A|\mathfrak{M}$ of A to the subspace $\mathfrak{M}$ is a bounded operator whose range is included in $\mathfrak{M}$. If Δ belongs to the set $\mathscr{I}$ of all finite subintervals of $(-\infty, \infty)$, then the set $S(A, \Delta)$ consists of all $\mathfrak{M} \in S(A)$ such that $\sigma(A|\mathfrak{M})$ is included in the closure of Δ. The operator A is called an "S-operator" if there exists a mapping E of $\mathscr{I}$ into $S(A)$ whose range is complete (as a Boolean algebra) and such that the following two conditions are satisfied for any $\Delta \in \mathscr{I}$: (1) any member of $S(A, \Delta)$ is a subset of $E(\Delta)$, (2) there is an interval I such that closure $(I)=$ closure (Δ), and $\sigma(A|E(\Delta))=\sigma(A) \cap I$.

The authors announce several sufficient conditions for A to be an S-operator; the conditions are carefully discussed, but no proofs are given. In particular, it is announced that the finiteness of the integral

$$\int_0^\infty \log \log \sup\{\|R(A, \lambda)\| : |\operatorname{Im} \lambda| \geqq t\} dt$$

implies that A is an S-operator (here $R(A, \lambda)=$ the value of the resolvent of A at λ). Let a be an arbitrary element of the domain $\mathscr{D}(A)$ of A, and suppose that there is a unique strongly differentiable function $\{t \to f(A, a)_t\}$ with values in $\mathscr{D}(A)$ which satisfies the following Cauchy problem: $f(A, a)_0=a$ and $A(f(A, a)_t)=i\cdot \partial f(A, a)_t/dt$, $t \in (-\infty, \infty)$. Under these circumstances, the operator $f(A, \cdot)_t=\{a \to f(A, a)_t\}$ may have a bounded norm $\|f(A, \cdot)_t\|$; the operator A is called "locally correct" if the function $\{t \to \|f(A, \cdot)_t\|\}$ is uniformly bounded in each finite subinterval of $(-\infty, \infty)$. Theorem: Suppose that A is locally correct; if the integral

$$\int_{-\infty}^{\infty} (1+t^2)^{-1} \log \|f(A, \cdot)_t\| dt$$

converges, then A is an S-operator.
G. L. Krabbe (New Haven, Conn.)

3981a:

Livšic, M. S. On a class of linear operators in Hilbert space. Amer. Math. Soc. Transl. (2) **13** (1960), 61–83.

3981b:

Livšic, M. S. Isometric operators with equal deficiency indices, quasi-unitary operators. Amer. Math. Soc. Transl. (2) **13** (1960), 85–103.

The Russian originals of these papers [Mat. Sb. (N.S.) **19 (61)** (1946), 239–262; **26 (68)** (1950), 247–264] have been reviewed [MR **8**, 588; **11**, 669], as has the preliminary announcement of the second [Dokl. Akad. Nauk SSSR (N.S.) **58** (1947), 13–15; MR **9**, 446].

3982:

Brodskiĭ, M. S.; Livšic, M. S. Spectral analysis of non-selfadjoint operators and intermediate systems. Amer. Math. Soc. Transl. (2) **13** (1960), 265–346.

The Russian original [Uspehi Mat. Nauk (N.S.) **13** (1958), no. 1 (79), 3–85] has already been reviewed [MR **20** #7221].

3983:

Iohvidov, I. S.; Kreĭn, M. G. Spectral theory of operators in spaces with an indefinite metric. I. Amer. Math. Soc. Transl. (2) **13** (1960), 105–175.

The Russian original [Trudy Moskov. Mat. Obšč. **5** (1956), 367–432] has been reviewed [MR **18**, 320], as has part 2 [ibid. **8** (1959), 413–496; MR **21** #6543].

3984:

Gohberg, I. C.; Kreĭn, M. G. The basic propositions on defect numbers, root numbers and indices of linear operators. Amer. Math. Soc. Transl. (2) **13** (1960), 185–264.

Translation of a Russian paper [Uspehi Mat. Nauk (N.S.) **12** (1957), no. 2 (74), 43–118; MR **20** #3459]. The translator has added eleven items to the bibliography [cf. also T. Kato, J. Analyse. Math. **6** (1958), 261–322 [MR **21** #6541]].

3985:

Foiaş, C.; Gehér, L.; Sz.-Nagy, B. On the permutability condition of quantum mechanics. Acta Sci. Math. Szeged **21** (1960), 78–89.

The authors' starting point is the problem of finding all (self adjoint) solutions of the commutation relation $PQ-QP=-iI$. This problem was first solved by replacing

it by the (formally equivalent) problem of finding all pairs of one parameter unitary groups $U_t = \exp(itP)$ and $V_s = \exp(isQ)$ satisfying the commutation relation $\exp(itP)\exp(isQ) = \exp(its)\exp(isQ)\exp(itP)$. Later however a solution was given to the problem in its original form (Rellich with improvements by Dixmier). The main theorem of the present paper gives conditions under which each form of the commutation relation implies the other. Moreover this theorem is proved in a more general setting in which U and V are semigroups of contraction operators. The authors state that the generalization to the case corresponding to that in which there are several P's and Q's may be carried out in a straightforward manner. The proof makes use of a functional calculus for contraction operators developed earlier by two of the authors [Sz.-Nagy and Foiaş, same Acta **19** (1958), 26–45; MR **21** #2188]. *G. W. Mackey* (Cambridge, Mass.)

3986:

Butzer, P. L.; Tillmann, H. G. An approximation theorem for semi-groups of operators. Bull. Amer. Math. Soc. **66** (1960), 191–193.

Let X be a complex B-space, let $E(X)$ be the algebra of bounded linear transformations on X into itself, and let $\{T(t)\}$ be a one-parameter semi-group in $E(X)$ of class $(1, C_1)$ in the notation of Hille and Phillips [*Functional analysis and semi-groups*, Amer. Math. Soc., Providence, R.I., 1957; MR **19**, 664]. Let A be the infinitesimal operator of $\{T(t)\}$ and denote by $D(A^\nu)$ the domain of the iterated operator A^ν. The authors state and sketch a proof of the following result. Let $f_0 \in D(A^{\nu-1})$. If there exists a g_0 such that

$$(1)\qquad \liminf_{t\to+0} \left| \frac{\nu!}{t^\nu}[T(t) - \sum_{k=0}^{\nu-1} \frac{t^k}{k!} A^k]f_0 - g_0 \right| = 0,$$

then $f_0 \in D(A^\nu)$ and $A^\nu f_0 = g_0$. If this holds with $g_0 = 0$, then the quantity following the lim inf is identically 0. If X is reflexive and omitting g_0 in (1) gives a finite result, then $f_0 \in D(A^\nu)$ and

$$(2)\qquad \lim_{t\to+0} \frac{\nu!}{t^\nu}[T(t) - \sum_{k=0}^{\nu-1} \frac{t^k}{k!} A^k]f_0 = A^\nu f_0.$$

In any case, the quantity following the lim is $o(t^{-1})$.
E. Hille (New Haven, Conn.)

3987:

Gel'fand, I. M. The structure of a ring of rapidly decreasing functions on a Lie group. Dokl. Akad. Nauk SSSR **124** (1959), 19–21. (Russian)

Let Γ be the algebra of all rapidly decreasing indefinitely differentiable functions on the group $G = \mathrm{SL}(2, C)$ of complex 2×2 matrices of determinant one. The author outlines a description of the image of Γ under the Fourier transform on G. If $f \in \Gamma$ and U varies over the irreducible unitary representations of G, then $\int f(g)U(g)\,dg$ can be given by an integral operator, say $K(z_1, z_2, U)$, as follows from the explicit knowledge of all irreducible unitary representations of G; here z_1 and z_2 are complex variables. Thus $K(z_1, z_2, U)$ can be taken to be the Fourier transform of f. The author now gives a description of the image of Γ in terms of growth and regularity conditions on $K(z_1, z_2, U)$. No proofs are indicated, but the author says that his method of proof is different from that which L. Ehrenpreis and the reviewer employed in the real case [Ann. of Math. (2) **61** (1955), 406–439; Trans. Amer. Math. Soc. **84** (1957), 1–55; **90** (1959), 431–484; MR **16**, 1017; **18**, 745; **21** #1541].
F. I. Mautner (Baltimore, Md.)

3988:

Želobenko, D. P. Structure of the group ring of the Lorentz group. Dokl. Akad. Nauk SSSR **126** (1959), 482–485. (Russian)

Let U denote the unitary subgroup of the group A of all 2×2 complex unimodular matrices. For each integer m and each complex number ρ let $S_{m,\rho}$ denote the (not necessarily unitary) Hilbert space representation defined by Naĭmark [same Dokl. **97** (1954), 969–972; MR **16**, 218]. Recall that all $S_{m,\rho}$ act in $L^2(U)$ and that $S_{m,\rho}$ is completely irreducible when ρ^2 is not of the form $-(m+2n)^2$ ($n = 1, 2, \cdots$). Let X be the ring under convolution of all infinitely differentiable complex valued functions on A with compact support. For each x in X the operator $\int S_{m,\rho}(y)x(y)dy = V_x{}^m$ is an integral operator with kernel $K_x(u_1, u_2, m, \rho)$. The main result announced in this note is a (somewhat complicated) necessary and sufficient condition that a function $K(u_1, u_2, m, \rho)$ be of the form K_x for some x in X. Closely related results have been announced by the author [ibid. **121** (1958), 586–589; MR **21** #2920] and by Gel'fand [see preceding review]. The corresponding question for the real 2×2 unimodular group has been treated by Ehrenpreis and Mautner [Trans. Amer. Math. Soc. **84** (1957), 1–55; MR **18**, 745].
G. W. Mackey (Cambridge, Mass.)

3989:

Wermer, John. On a subalgebra of $L(-\infty, \infty)$. Amer. J. Math. **82** (1960), 103–105.

Let L be the normed algebra of all Lebesgue integrable functions on the real line with product as convolution and norm defined by $\|f\| = \int_{-\infty}^{+\infty} |f(x)|dx$. Let L^+ be the closed subalgebra of all functions in L which vanish for negative x. Theorem: L^+ is a maximal closed subalgebra of L. This is proved by deducing it from the following result: Let B be a commutative normed algebra whose maximal ideal space is the unit circle $|z| = 1$, hence a function algebra on $|z| = 1$. Assume that the functions z and $1/z$ lie in B and generate a dense subalgebra of B. Let B^+ be the subalgebra of those functions in B which have continuous extensions to $|z| \leq 1$ which are analytic in $|z| < 1$. Then B^+ is a maximal subalgebra of B.
F. I. Mautner (Baltimore, Md.)

3990:

Bear, H. S. Some boundary properties of function algebras. Proc. Amer. Math. Soc. **11** (1960), 1–4.

Let X be a compact Hausdorff space, $C(X)$ $[C_R(X)]$ the algebra of all continuous complex [real] functions on X, A a closed subalgebra of $C(X)$ separating points and containing the constants. Let Σ be the maximal ideal space of A and assume X is a proper subset of Σ. A 'maximum set' is any set $\{x \in \Sigma \,|\, f(x) = \|f\|\}$ for some non-constant f in A. Assume that A contains no proper closed ideal of $C(X)$. Theorem 1: If E is a maximum set, then the restriction $A|E$ of A to E is a closed subalgebra of $C(E)$. Theorem 2: If E is a closed subset of Σ with $A|E = C(E)$, and $x_0 \in \Sigma - E$, then there exists $f \in A$ with $f = 0$ on E and $f(x_0) \neq 0$. Theorem 3: If E is a maximum set for A contained in X,

then $m(E)=0$ for every representing measure m. A positive measure m on X is called a 'representing measure' for the point x in Σ if $\int f\,dm=f(x)$ for all f in A. Theorem 4: Let the real parts of functions in A be uniformly dense in $C_R(X)$. If m represents x_0 in $\Sigma-X$ and $m(E)\neq 0$, then if $f\in A$ and $f=0$ on E, we get $f(x_0)=0$. Various corollaries of these results are also given.

J. Wermer (Providence, R.I.)

3991:

Takesaki, Masamichi. On the singularity of a positive linear functional on operator algebra. Proc. Japan Acad. **35** (1959), 365–366.

A positive linear functional on a W^* algebra has been called 'singular' if it dominates no non-zero weakly continuous linear functional. The following new characterization is given: a positive linear functional φ is singular if and only if, for every non-zero projection e, there is a non-zero projection $f\leqq e$ with $\varphi(f)=0$. This enables the author to establish more simply his decomposition of a positive linear faithful mapping π of one W^* algebra M into another N, into a weakly continuous part and a singular part.

J. Feldman (Berkeley, Calif.)

3992:

Pukánszky, Lajos. On maximal abelian subrings of factors of type II_1. Canad. J. Math. **12** (1960), 289–296.

The author sets up certain invariants for a maximal abelian subring P of a type II_1 factor, relative to algebraic isomorphism, which are relatively computable—specifically, the invariants relative to unitary equivalence of the (abelian) ring generated by left and right multiplications by elements of P acting on M formulated as a pre-Hilbert space relative to the inner product determined by the trace. This together with a detailed development of certain examples of II_1 rings similar to ones treated by Murray and von Neumann enable him to show that there exist infinitely many 'singular' maximal abelian subrings of an approximately finite II_1 factor which are inequivalent under automorphisms of the factor—a singular subring being one with the property that every automorphism of the ring leaving the subring invariant is implementable by a unitary element of the subring. The apparent complexity of the equivalence problem for maximal abelian subrings of II_1 factors relative to the same problem for I_∞ factors, treated by Dixmier [Ann. of Math. (2) **59** (1954), 279–286; MR **15**, 539], is thus both amplified and clarified. Whether the present invariants form a complete set for singular maximal abelian subrings of an approximately finite II_1 factor is left open.

I. E. Segal (Cambridge, Mass.)

3993:

Gohberg, I. C. On bounds of indexes of matrix-functions. Uspehi Mat. Nauk **14** (1959), no. 4 (88), 159–163. (Russian)

After a brief survey of the theory of a "left (or right) standard factorisation", appended to a previous article of the author and M. G. Kreĭn [same Uspehi **13** (1958), no. 2 (80), 3–72; MR **21** #1506], the following theorem is proved. Let $\mathfrak{A}(\zeta)=\sum_{j=-\infty}^{\infty}A_j\zeta^j\in\mathfrak{R}$ be a matrix-function, with $\det\mathfrak{A}(\zeta)\neq 0$, $|\zeta|=1$; and let $\mathfrak{A}^{-1}(\zeta)\in\mathfrak{R}$ be such that $\mathfrak{A}^{-1}(\zeta)=\sum_{j=-\infty}^{\infty}B_j\zeta^j$ ($|\zeta|=1$). If the inequalities

$$\sum_{j=-\infty}^{-q-1}\|A_j\|<\Big(\sum_{j=-\infty}^{\infty}\|B_j\|\Big)^{-1},\quad \sum_{j=p+1}^{\infty}\|A_j\|<\Big(\sum_{j=-\infty}^{\infty}\|B_j\|\Big)^{-1}$$

are valid, where p and q are integers, then all right indices $\kappa_j(\mathfrak{A})$ of the matrix-function $\mathfrak{A}(\zeta)$ are bounded by $q\leqq\kappa_j(\mathfrak{A})\leqq p$ ($j=1,2,\cdots,n$).

D. Mangeron (Iaşi)

3994:

Nečepurenko, M. I. On implicit functions. Leningrad. Gos. Univ. Uč. Zap. Ser. Mat. Nauk **33** (1958), 32–36. (Russian)

L. V. Kantorovič [Dokl. Akad. Nauk SSSR **76** (1951), 17–20; **80** (1951), 849–852; MR **12**, 835; **13**, 469] considered the use of Newton's method for the location of roots of functions with domain in a vector space which is "normed" by the elements of a partially ordered vector space and where the function to be solved is suitably majorized. The present author considers implicit functions in a similar vein. Let X, Λ, and Y be normed by three partially ordered vector spaces and let $P:X\times\Lambda\to Y$ be a function with $P(x_0,\lambda_0)=0$, and consider the problem of finding local solutions $x=x(\lambda)$ of $P(x,\lambda)=0$ near λ_0. It is assumed that P is analytic in an appropriate sense near (x_0,λ_0), that the partial derivative $P_x'(x_0,\lambda_0)$ has a bounded inverse and that P is majorized by a suitable function in the associated p.o. spaces. Specific hypotheses (which cannot be reproduced here) assure the existence and the local uniqueness of the desired solutions.

R. G. Bartle (Urbana, Ill.)

3995:

Iino, Riichi. Sur les dérivations dans les espaces vectoriels topologiques sur le corps des nombres complexes. I, II, III. Proc. Japan Acad. **35** (1959), 343–348, 530–535; **36** (1960), 27–32.

Let E and F be separated and complete locally convex topological vector spaces over the complex field C. In I, the author introduces the concepts of Fréchet differentials and of weak (or directional) derivatives of functions f with domain in E and range in F. The first Fréchet differential of f at x_0 is denoted by $df(x_0)$: it is characterised (when it exists) as that element T of $\mathscr{L}(E,F)$, the set of continuous linear maps of E into F, such that $f(x_0+h)-f(x_0)=T(h)+\alpha(x_0;h)$, where $\alpha(x_0;h)=o(h)$ as $h\to 0$. A function g mapping a punctured neighbourhood of 0 in E into F is said to be $o(x^n)$ as $x\to 0$ if, for each continuous seminorm q on F, there exists a continuous seminorm p on E and a numerical function ε such that $\varepsilon(x)\to 0$ as $x\to 0$ and $q(g(x))\leqq\varepsilon(x)[p(x)]^n$ as $x\to 0$. The map $f':x\to df(x)$, with domain the set of all x for which $df(x)$ exists, is called the first derivative of f. Weak derivatives of f at x_0 are defined in terms of the functions $\varphi(t)=f(x_0+th)$, t being a complex variable: $\varphi'(0)$, if it exists, is the weak derivative of f at x_0 in the direction h and is denoted by $Df(x_0)[h]$. If this is the case for all h, then $h\to Df(x_0)[h]$ is linear from E into F, but not necessarily continuous. The existence of $df(x_0)$ implies that of $Df(x_0)[h]$ for all h, in which case $df(x_0)=Df(x_0)$. A mean value theorem is established for functions for which $Df(x_0)[h]$ exists for a given x_0 and h. This is used to show that if $Df(x)[h]$ exists for all h and all x in a neighbourhood U of x_0, if $Df(x)$ is continuous for each x in U, and if $x\to Df(x)$ is continuous from U into $\mathscr{L}_b(E,F)$ $(=\mathscr{L}(E,F)$ equipped with the topology of convergence uniform on

the bounded subsets of E), then $df(x_0)$ exists and equals $Df(x_0)$.

In II, monomials of degree n, denoted by $a_n x^n$, are introduced as functions of the form $x \to a_n(x, x, \cdots, x)$, where a_n is a continuous symmetric n-linear function from E^n into F. Each such monomial is Fréchet differentiable at all points of E. A polynomial is a finite linear combination of monomials. If f maps a neighbourhood of x_0 in E into F, and if there exists a polynomial P_n of degree at most n ($n \geq 1$) such that $f(x_0+h)-f(x_0)=P_n(h)+\alpha_n(x_0;h)$, where $\alpha_n(x_0;h)=o(h^n)$ as $h\to 0$, then f is said to be n times Fréchet differentiable at x_0, and $d^nf(x_0): h\to a_nh^n$ (the monomial of degree n appearing in P_n) is the nth Fréchet differential of f at x_0. If $d^nf(x_0)$ exists, the polynomial P_n is necessarily given by the formula $P_n(h)=\sum_{0\leq m\leq n} d^mf(x_0)[h]/m!$ and we arrive at a Taylor development of order n. When F is a topological algebra, Leibnitz' formula is established for functions f and g for which $d^nf(x_0)$ and $d^ng(x_0)$ exist. Weak derivatives $D^nf(x_0)$ of order $n\geq 1$ are defined via the functions $\varphi(t)=f(x_0+th)$ as

$$D^nf(x_0)[h] = D^nf(x_0;h) = \varphi^{(n)}(0).$$

$D^nf(x_0)$ is the monomial of degree n derived from the symmetric n-linear function defined by

$$(h_1, \cdots, h_n) \to D^nf(x_0)[h_1, \cdots, h_n] = \left[\frac{\partial^n}{\partial t_1 \cdots \partial t_n} f(x_0+t_1h_1+\cdots+t_nh_n)\right]_{t_1=\cdots=t_n=0}.$$

Relations between this concept and the nth order Fréchet differential (analogous to those for $n=1$) are given.

In III, it is shown that an f which is continuous and weakly differentiable in all directions h at all points x of a domain $\Omega \subset E$ is completely determined by its values in any non-void open subset Ω_1 of Ω. A version of Liouville's theorem is also given. $D(\Omega, F)$ is introduced as the vector space of functions f of the type described in the penultimate sentence. This space may be topologised in various ways, all considered here being defined by convergence uniform on the sets of certain families $\mathfrak{S}$ of subsets of Ω, and all making $D(\Omega, F)$ into an additive topological group but not always a topological vector space. The completeness of $D(\Omega, F)$ for such structures is considered, as also is the continuity of $f\to f'$ in various senses. *R. E. Edwards* (Reading)

3996:

Slugin, S. N. Iterational method of one-sided approximations of the solution of operator equations. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 117–124. (Russian)

Let X be a partially ordered space, and $\underline{x}_0, \bar{x}_0 \in X$, $\underline{x}_0 \leq \bar{x}_0$. Let Γ and Λ be positive additive operators from X to X; let $(I-\Lambda)^{-1}$ exist, and for $x\geq 0$ let

$$\lim_{n\to\infty} (\Gamma+2\Lambda)^n x = 0.$$

Further let the monotone continuous operator V from X to X satisfy the conditions $\Lambda(\Delta x)\geq V(x+\Delta x)-V(x)\geq -\Gamma(\Delta x)$ for $\Delta x\geq 0$, $\Lambda(\bar{x}_0-\underline{x}_0)\leq V(\bar{x}_0-\underline{x}_0)$, $\Lambda(\bar{x}_0-\underline{x}_0)\leq \bar{x}_0-V(\underline{x}_0)$. The following theorem is proved: the equation $x=V(x)$ has a unique solution, to which converges monotonely the sequence defined recursively by $x_{n+1}=x_n-(I-\Lambda)^{-1}(x_n)$, $x_0=\bar{x}_0$ or $\underline{x}_0$. The theorem is applied to the Cauchy problem for an ordinary differential equation, to a system of first-order differential equations, to a differential equation with lagging argument, and to nonlinear integral equations. *A. N. Baluev* (RŽMat 1957 #8253)

3997:

Zuhovickiĭ, S. I.; Èskin, G. I. Some theorems on best approximation by unbounded operator functions. Izv. Akad. Nauk SSSR. Ser. Mat. **24** (1960), 93–102. (Russian)

Complete proofs and extensions to reflexive Banach spaces of results announced previously [Dokl. Akad. Nauk SSSR **116** (1957), 731–734; MR **20** #2610]. See also S. B. Stečkin [Rev. Math. Pures Appl. **1** (1956), no. 3, 79–83; MR **20** #6002]. *E. Hewitt* (Seattle, Wash.)

CALCULUS OF VARIATIONS

3998:

Forsyth, A. R. ★Calculus of variations. Dover Publications, Inc., New York, 1960. xxii+656 pp. $2.95.

Unabridged and unaltered republication of the Cambridge University Press edition, 1927.

3999:

Sigalov, A. G. Variational problems with admissible surfaces of arbitrary topological types. Amer. Math. Soc. Transl. (2) **14** (1960), 59–106.

Translation of Uspehi Mat. Nauk **12** (1957), no. 1 (73), 53–98 [MR **19**, 560].

4000:

Sigalov, A. G. Amendment to the article "Variational problems with permissible surfaces of arbitrary topological types". Uspehi Mat. Nauk **15** (1960), no. 1 (91), 261. (Russian)

This brief amendment to the author's article [see preceding review] is intended to dispose of a crucial oversight, pointed out by W. H. Fleming and noted by the reviewer in MR **19**, 560. The matter could have been settled by appealing to powerful methods and results of surface area, or to recent work by W. H. Fleming, if this did not run counter to the whole spirit of the very beautiful Sigalov approach. After studying the author's amendment, the reviewer could only form the impression, from its scanty indications, that it underestimates the difficulty of carrying out the constructions which are proposed in it. And yet it seems plausible that somewhat similar constructions may provide what is needed.

L. C. Young (Bloomington, Ind.)

4001:

Kósa, A. Notwendige Bedingungen für die diskontinuierlichen Lösungen von den Variationsproblemen n-ter Ordnung. Acta Math. Acad. Sci. Hungar. **11** (1960), 23–48. (Russian summary, unbound insert)

The problem considered is that of minimizing an integral

$$I[y] = \int_{x_1}^{x_2} f(x, y, y', \cdots, y^{(n)})\, dx$$

in a class of continuous curves $y=y(x)$ which have continuous derivatives up to and including the nth order, except at a finite number of points in the interval (x_1, x_2), at which all these derivatives have one-sided limits. The curves are also required to satisfy end-conditions $y(x_1)=y_1$, $y(x_2)=y_2$, and

$$(*)\quad y^{(i)}(x_1) = y_1^{(i)},\quad y^{(i)}(x_2) = y_2^{(i)}\quad (i = 1, \cdots, n-1).$$

The necessary conditions derived include differential equations of the expected form, corner conditions, and extensions of the conditions of Legendre and of Weierstrass. The end conditions (*) are from the nature of the problem spurious, since discontinuities in the derivatives are permitted arbitrarily close to the ends of the interval (x_1, x_2). Hence (*) will be satisfied by a minimizing curve only if the values $y_1^{(i)}$, $y_2^{(i)}$ chosen are those enforced by the nature of the problem. There are n differential equations to be satisfied by a minimizing curve, so these differential equations are usually inconsistent, and the problem has no solution. However, a number of examples are given in which a solution exists.

L. M. Graves (Chicago, Ill.)

4002:

Ingarden, R. S. Composite variational problems. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 687–689. (Russian summary, unbound insert)

The author shows that the usual variational techniques can be applied to the problem of finding the extrema of functionals of the form $U(f)=\int_0^a F(t, f, V(f))dt$, where $V(f)$ is itself an integral functional of f.

R. E. Bellman (Santa Monica, Calif.)

4003:

Gurevič, A. S. On degenerate problems in the calculus of variations. Vestnik Leningrad. Univ. **14** (1959), no. 19, 64–77. (Russian. English summary)

Let $F(x, y, z)$ be a thrice differentiable real function defined in a domain R_1, where x denotes a real variable, and y, z variable vectors with n components. Further let $g(x_1, y_1; x_2, y_2)$ be a thrice differentiable vector function, with g ($\leq 2n+2$) components, whose arguments are a pair of positions of (x, y), and suppose g defined in a domain R_2 of the relevant product space. A continuously differentiable non-parametric curve $y(x)$ $(x_1 \leq x \leq x_2)$ is admissible if its pair of ends belongs to R_2 and if, for each x of $(x_1 \leq x \leq x_2)$, the triple $x, y(x), y'(x)$ belongs to R_1; it is an extremal if it annuls the first variation of the integral of F. The main theorem asserts that, for an extremal $y(x)$, if the Hessian in z of F has rank r at a, $y(a)$, $y'(a)$ where $x_1 < a < x_2$, then at least r components of the function $y(x)$ have continuous first derivatives near $x=a$.

L. C. Young (Bloomington, Ind.)

GEOMETRY

See also 4044.

4004:

Bachmann, Friedrich; Pejas, Wolfgang. Metrische Teilebenen hyperbolischer projektiv-metrischer Ebenen. Math. Ann. **140** (1960), 1–8.

The authors call a metric plane a system formed by a set of "points", a set of "lines", an incidence relation of point and line, and a perpendicularity relation of lines provided that (1) corresponding to each two distinct points there is a unique line incident with them both, (2) there are at least two points, (3) if line b is perpendicular to line c, then c is perpendicular to b, (4) perpendicular lines are incident with a common point, (5) if p is a point and b is a line there is line incident with p that is perpendicular to b, and if p is incident with b, only one such line. An earlier work [F. Bachmann, *Aufbau der Geometrie aus dem Spiegelungsbegriff*, Springer-Verlag, Berlin, 1959; MR **21** #6557] has left unanswered the question whether each such metric plane is a sub-plane of a euclidean, elliptic or hyperbolic plane, where these concepts are used in a sense defined in the work cited above. The present article supplies a negative answer to the question.

L. M. Blumenthal (Columbia, Mo.)

4005:

Filippov, P. V. On some rules of connection in central and parallel projecting. Zap. Leningrad. Gorn. Inst. **36** (1958), no. 3, 156–165. (Russian)

4006:

Filippov, P. V. Conjugated projections. Zap. Leningrad. Gorn. Inst. **36** (1958), no. 3, 166–183. (Russian)

4007:

da Silva Lobo, Hamilcar. On determination of the distance from a point to a straight line in the general case of oblique ones by a change of the system of reference. Ciência. Lisboa No. 15/16 (1958/59), 71–75. (Portuguese)

4008:

Molnár, F.; Molnár, J. Über eine Verallgemeinerung der Tschirnhausschen Flächen und Kurven. Acta Math. Acad. Sci. Hungar. **10** (1959), 269–276. (Russian summary, unbound insert)

Let r_i be the distances from a variable point P in euclidean space to h given fixed points F_i (foci), d_j the distances from P to m given planes σ_j (focal planes) and e_k the distances from P to n given lines g_k (focal lines). The authors study the geometrical locus of the points P for which an equation of the form

$$\sum_1^h \rho_i r_i + \sum_1^m \delta_j d_j + \sum_1^n \varepsilon_k e_k - C = 0$$

holds, where ρ_i, δ_j, ε_k, C are given constants. These loci are called T-surfaces (T-lines if $m=0$ and the h points and n lines are in a plane); they generalize the surfaces of Tschirnhaus considered by Gy. Sz.-Nagy [same Acta **1** (1950), 36–45, 167–181; **5** (1954), 165–167; MR **12**, 733; **13**, 983; **16**, 1046]. In particular, the points at infinity, the normal lines, the convexity properties and the existence of a principal sphere are discussed.

L. A. Santaló (Buenos Aires)

4009:

Zobel, A. Intersection theory on an open variety. Ann. Mat. Pura Appl. (4) **46** (1958), 1–17.

Severi nella Memoria [Ann. Mat. Pura Appl. (4) **19**

(1940), 153–242; MR 7, 476] ha considerato due metodi per ottenere una teoria dell'equivalenza algebrica e una teoria delle intersezioni su una varietà algebrica aperta, cioè una varietà algebrica W da cui sia stata tolta una sua sottovarietà S. Il primo metodo è basato su una certa definizione di equivalenza su $W-S$ fra sottovarietà di W. L'A. osserva che tale metodo conduce ad una classificazione delle sottovarietà di W rispetto a $W-S$ (W ed S non singolari) piuttosto che ad una soddisfacente teoria delle intersezioni.

A tale scopo si presta meglio il secondo metodo, soltanto accennato da Severi e qui sviluppato sotto ipotesi abbastanza generali. Sia φ una dilatazione di W di base S e sia $*W=\varphi(W)$ non singolare. Detta sottovarietà di $W-S$ una sottovarietà di W priva di componenti su S, l'equivalenza algebrica e l'intersezione virtuale di sottovarietà di $W-S$ si definiscono ora passando alle loro trasformate (nel senso di Hodge-Pedoe) su $*W$.

Questo secondo metodo viene infine applicato alla costruzione esplicita delle basi algebriche e delle loro intersezioni su $W-S$ quando W è un cono senza altre singolarità fuori del suo vertice S.

F. Gherardelli (Florence)

4010:

Morgantini, Edmondo. Sulla configurazione di tre omologie piane in posizione omologica. Rend. Sem. Mat. Univ. Padova **29** (1959), 328–400.

"In an equilateral triangle the three points P_1, P_2, P_3, which are symmetrical to a point P with respect to the three sides, are the vertices of a triangle homologous to the given one." This theorem has led the author to the investigation of those triples of plane homologies Ω_i, $i=1, 2, 3$, which are in punctual homology, and also of the dual case in which the triangle of sides g_1, g_2, g_3 homologous to a line g is always homologous to the triangle A_1, A_2, A_3 of the centers of Ω_i. The points A_i are taken as fundamental points of a coordinate system so that any homology with center A_1 can be given by $y_1:y_2:y_3=(a_1x_1+a_2x_2+a_3x_3):x_2:x_3$. Then follows a careful discussion of the general cases and of special ones, and the construction of affine and metric models.

D. J. Struik (Cambridge, Mass.)

4011:

Killgrove, Raymond B. A note on the nonexistence of certain projective planes of order nine. Math. Comput. **14** (1960), 70–71.

A search carried out on SWAC shows that there is no plane of order 9 which possesses an additive loop which is the cyclic group of order 9.

Marshall Hall, Jr. (Pasadena, Calif.)

4012:

Martynenko, V. S. On cases of coincidence of the hyperbolic magnitude of an angle in the Lobačevskiĭ plane with its euclidean magnitude in the Beltrami model. Ukrain. Mat. Ž. **11** (1959), 109–110. (Russian)

Let B be the Beltrami (or Cayley-Klein) model of the Lobačevskiĭ plane (i.e., the euclidean disk of radius 1 and center O, in which the non-euclidean straight lines coincide with the euclidean ones), and let O' be a point in B, different from O. For each point P in the euclidean plane, a pair of rays $r_1(P)$, $r_2(P)$ and two angles $\varphi(P)$ and $\varphi_e(P)$ are defined as follows. If $\overline{O'P}$ is the euclidean length of the segment $O'P$, then $\varphi_e(P)=\arctan(\overline{O'P})$. The rays $r_1(P)$, $r_2(P)$ span the euclidean angle $\varphi_e(P)$, with vertex O'; and the euclidean bisector of this angle is $O'P$. $\varphi(P)$ is the non-euclidean angle spanned by $r_1(P)$ and $r_2(P)$.

The author considers the set $C=\{P \mid \varphi(P)=\varphi_e(P)\}$. It is shown that C is a curve, with equation (relative to a cartesian coordinate system with origin O' and x-axis OO')

$$(x^2-y^2)^2[(1+m)^2(x^2+y^2)+4m]-4(1-m)^2x^2y^2 = 0,$$

where $\sqrt{(1-m^2)}$ is the euclidean length $\overline{OO'}$.

L. Greenberg (Copenhagen)

CONVEX SETS AND GEOMETRIC INEQUALITIES

4013:

Dubins, Lester E. Another proof of the four vertex theorem. Amer. Math. Monthly **67** (1960), 573–574.

Based on a theorem of Schur which says (roughly): If a convex arc has everywhere greater curvature than another then its chord is shorter. *P. Ungar* (New York)

4014:

Firey, W. J. Isoperimetric ratios of Reuleaux polygons. Pacific J. Math. **10** (1960), 823–829.

For any odd integer n, let Γ_n denote the class of Reuleaux polygons C_n of width 1 and having n sides, i.e., C_n is a set of constant width 1 in the Euclidean plane whose boundary consists of n circular arcs each of radius 1. The length of the boundary of any set of constant width d is πd [T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper*, Springer, Berlin, 1934]; so that a study of the isoperimetric ratio for Reuleaux polygons amounts to a study of the area, $A(C_n)$, of the figures $C_n \in \Gamma_n$. If C_n' denotes the regular $C_n \in \Gamma_n$, the author proves: (i) $A(C_3')<A(C_5')<\cdots<A(C'_{2k+1})<\cdots$; (ii) $\sup_{C_n\in\Gamma_n} A(C_n)=A(C_n')$, and the extremal curve is unique; (iii) $\inf_{C_n\in\Gamma_n} A(C_n)=A(C_3')$, and the infimum is not attained.

S. J. Taylor (Ithaca, N.Y.)

4015:

Woods, A. C. On the irreducibility of convex bodies. Canad. J. Math. **11** (1959), 256–261.

The first result is similar to results of Mahler [Nederl. Akad. Wetensch. Proc. **49** (1946), 331–343; MR **8**, 12] and Rogers [ibid. **50** (1947), 868–872; MR **9**, 228] for irreducible star bodies. An example shows that a convex body may be irreducible among the convex bodies, but not among the star bodies. The terminology "the lattice Λ is free at the point X" is not elegant, as in the present paper it is a qualification of X, but not of Λ.

C. G. Lekkerkerker (Amsterdam)

4016:

Fejes Tóth, L. Sur la représentation d'une population infinie par un nombre fini d'éléments. Acta Math. Acad. Sci. Hungar. **10** (1959), 299–304. (Russian summary, unbound insert)

The author proves that if a plane disk with density $f(x, y)$ is divided into a finite number of disks whose sum of moments of inertia about centroids is k and if for all

possible sub-divisions into n parts the minimal value of k is S_n, then

$$\lim_{n\to\infty} nS_n = \frac{5\sqrt{3}}{54}\left(\iint \sqrt{f(x, y)}\, dxdy\right)^2,$$

where the integral is over the original disk.

H. G. Eggleston (Cambridge, England)

4017:

Fejes Tóth, L. Verdeckung einer Kugel durch Kugeln. Publ. Math. Debrecen **6** (1959), 234–240.

The author considers various arrangements of solid (non-overlapping) spheres in Euclidean space. Let $M(r)$ denote the smallest numbers of unit spheres that can be placed around a sphere K of radius r so that every ray from the center of K will intersect (not merely touch) at least one of the unit spheres. He gives an inequality for $M(r)$ which yields $M(1)\geq 19$ and $M(0)\geq 4$. He also proves $M(0)\leq 6$; thus $M(0)=4$, 5, or 6. Another problem about such a 'cloud' of unit spheres concerns its 'thickness' $R-r$, where R is the radius of the smallest sphere that encloses them all. This remains significant when r tends to infinity, so that we are considering unit spheres between two parallel planes such that every line perpendicular to the planes penetrates at least one of the spheres. Here the thickness attains its lower bound $2+\sqrt{2}$ when the centers of the spheres are at the vertices of reciprocal tessellations of squares in two parallel planes (between the two given planes). He leaves the reader to find the minimum thickness when the words 'perpendicular to the planes' are replaced by 'intersecting the planes'.

H. S. M. Coxeter (Toronto)

DIFFERENTIAL GEOMETRY

See also 4057.

4018:

Schwartz, Manuel; Green, Simon; Rutledge, W. A. ★Vector analysis: with applications to geometry and physics. Harper's Mathematics Series. Harper & Brothers, New York, 1960. xii+556 pp. $7.50.

A beginning text. From the preface: "It is hoped that the numerous examples and exercises will make the book valuable to a large number of engineers for self-study. The study of applications, that is, the relations among vector analysis, geometry, and physics, is carried along with the mathematical theory." Among the chapter headings are Differential geometry, Harmonic functions, and Magnetism and Electrodynamics.

4019:

Bonferroni, Carlo. La densità di probabilità come flusso di probabilità. Ann. Mat. Pura Appl. (4) **48** (1959), 387–402.

Betrachtet wird z.B. das Volumen $V(u)$ eines Bereichs $B(u)$, der von einer geschlossenen Fläche $\Sigma(u)$ begrenzt wird, die von einem Parameter u abhängt, so daß $B(u)$ dabei innerhalb eines festen Bereichs vom Volumen 1 variiert. Die Ableitung dV/du heißt dann der Wahrscheinlichkeitsfluß durch $\Sigma(u)$. Hierfür und für ähnliche Größen werden einige Formeln angegeben.

K. Krickeberg (Heidelberg)

4020:

Hsiao, E. K. Normal curvatures of a congruence of curves relative to a congruence. Tensor (N.S.) **10** (1960), 73–80.

Let S be a surface in Euclidean space E^3. Let P be a point of a curve C on S. Let v be a unit vector field in S. Let N be the unit normal vector to S in E^3. The normal curvature of v with respect to C, that is, the component along N of the derived vector of v with respect to C at P, was first investigated by the reviewer [Amer. J. Math. **74** (1952), 955–966; MR **14**, 406]. This notion was generalized by R. S. Mishra and S. Krishna [Tensor (N.S.) **6** (1956), 125–131; MR **18**, 761] who replaced v by a congruence and by R. N. Kaul [ibid. **7** (1957), 110–116; MR **19**, 1074] who replaced N by a congruence. Replacing both v and N by two congruences, the author of this paper gives another generalization. The reviewer doubts that the term normal curvature is appropriate for this paper.

T. K. Pan (Norman, Okla.)

4021:

Matsumoto, Makoto. Determination of the second fundamental form of a hypersurface by its mean curvature. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **32** (1959), 259–279.

T. Y. Thomas has shown [Bull. Amer. Math. Soc. **51** (1945), 390–399; MR **7**, 30] that the second fundamental form of a surface in E^3 is determined by its first fundamental form and mean curvature. The author derives Thomas' results using tensorial methods, and proceeds to find similar theorems for hypersurfaces of arbitrary dimension.

C. B. Allendoerfer (Seattle, Wash.)

4022:

Singal, M. K.; Behari, Ram. A note on a family of parallel hypersurfaces in a Riemannian V_n. Math. Student **25** (1957), 19–20.

Die Verff. beweisen den folgenden Satz: Die Hyperflächen einer Schar von Parallel-Hyperflächen sind genau dann zueinander isometrisch, wenn auf jeder einzelnen Hyperfläche der Schar die mittlere Krümmung konstant ist.

K. P. Grotemeyer (Zbl. **79**, 381)

4023:

Vranceanu, G. Sur la géométrie intrinsèque des espaces non holonomes. J. Math. Pures Appl. (9) **37** (1958), 1–19.

Let $ds^a=\lambda_i^a dx^i$ $(a, i=1, \cdots, n)$ be n independent forms on some space V_n $(|\lambda_i^a|\neq 0)$, and V_n^m be some nonholonomic subspace of V_n defined by the equations (1) $ds^\alpha=0$ $(\alpha=m+1, \cdots, n)$. Given on V_n a group of substitutions of linear forms:

$$(2)\qquad \begin{aligned} d\bar{s}^h &= c_k{}^h ds^k + c_\alpha{}^h ds^\alpha \qquad (h, k, l = 1, \cdots, m),\\ d\bar{s}^\alpha &= c_\beta{}^\alpha ds^\beta \qquad (\alpha, \beta, \gamma = m+1, \cdots, n).\end{aligned}$$

Here $c_k{}^h$, $c_\alpha{}^h$, $c_\beta{}^\alpha$ are functions of $x^1, \cdots, x^n$, for which we have $|c_\beta{}^\alpha|\neq 0$ and the orthogonality condition $c_l{}^h c_k{}^k=\delta_l{}^k$. Distinguish three sorts of properties of the space $V_n{}^m$: (i) the intrinsic, which are invariants of the group (2); (ii) the semi-intrinsic, which are invariants of the group (2) under the condition $c_\alpha{}^h=0$; (iii) the rigid, which are invariants of (2) under the conditions $c_\alpha{}^h=0$, $c_\alpha{}^\beta c_\beta{}^\gamma=\delta_\alpha{}^\gamma$.

In 1936 the author proved that for non-holonomic hypersurfaces the study of intrinsic properties can always be reduced to the study of rigid properties. This assertion is proved anew in the article under review. In 1946 M. Haimovici proved that if system (1) is of maximal rank and has no derived system (see below), then the study of intrinsic properties of a non-holonomic subspace V_n^m $(m<n-1)$ can also be reduced to the study of its rigid properties. The author gives in chapter 1 a new proof of Haimovici's assertion, based on a theorem of his own on singular Riemannian spaces, proved in 1942. If the equation in the second line of (2) can be put in the form

$$d\bar{s}^{\alpha} = c_{\beta}{}^{\alpha} ds^{\beta} + c_{\sigma}{}^{\alpha} ds^{\sigma} \qquad (\alpha, \beta = m+1, \cdots, q),$$

$$d\bar{s}^{\rho} = c_{\lambda}{}^{\rho} ds^{\lambda} \qquad (\rho, \lambda, \sigma = q+1, \cdots, n),$$

this means that system (1) has the derived system (S') $ds^{\rho}=0$. Similarly one may speak of derived systems S'', S''', etc. The author proves that if system (1) admits only a first derived system S' and both systems are of maximal rank, then group (2) can be reduced to the orthogonal group, hence the study of intrinsic properties of the non-holonomic space V_n^m can be reduced to the study of its rigid properties. If the number of derived systems is large, the author shows how to adjoin so-called invariant systems, and shows how to reduce the study of intrinsic properties of V_n^m to that of its rigid properties in the case that all invariant systems are of maximal rank and system (1) has no integrable combinations. The conditions the author uses are sufficient but not necessary. In conclusion, two examples are considered in which the conditions are not fulfilled but the reduction of intrinsic to rigid properties is possible.

N. I. Kovancov (RŽMat **1959** #11538)

4024:

Yano, Kentaro. Champs de vecteurs dans un espace riemannien ou hermitien. C. R. Acad. Sci. Paris **251** (1960), 194–195.

The report consists of three parts. (1) A theorem on the Lie derivative of a harmonic tensor with respect to a conformal Killing vector is given to generalize a former theorem on one with respect to a Killing vector [Yano, Ann. of Math. (2) **55** (1952), 38–45; MR **13**, 689]. (2) Let v^h be an infinitesimal transformation and $\xi^h(s)$ be a geodesic in V_n, a compact orientable Riemannian manifold of dimension n and of class C^{∞}. v^h carries $\xi^h(s)$ to a geodesic without changing the affine character of its arc length s, if and only if a certain vector vanishes. In that vector, replace $d\xi^h/ds$ by a unit vector λ^h and call the result the geodesic deviation vector of λ^h with respect to v^h. When the mean of geodesic deviation vectors of n mutually orthogonal unit vectors with respect to v^h vanishes, then v^h is defined as a geodesic vector in V_n. Several properties are given relative to the integral of the Ricci curvature $K_{ji}v^jv^i$, the existence of v^h in a V_n with negative definite or non-positive Ricci curvature $K_{ji}v^jv^i$ and the divergence of v^h in an Einstein V_n. (3) There are given necessary and sufficient conditions for a vector v^h to be almost analytic in almost Hermitian and Kaehler manifolds and sufficient conditions for a conformal Killing vector to define an automorphism of these manifolds.

T. K. Pan (Norman, Okla.)

4025:

Su, Buchin. A note on the theory of conjugate nets in hyperspace. Sci. Record (N.S.) **3** (1959), 441–445.

The author proves the following theorems. (1) In S_n, if a conjugate net is conjugate to a rectilinear congruence, then the Laplace sequence to which the net belongs must be inscribed in the Laplace sequence containing the focal nets of the congruence. Two Laplace sequences inscribed in the same Laplace sequence of the congruence have a harmonic Laplace sequence in common. (2) In an S_5 with Klein hyperquadric Q_4, represent the Godeaux sequence of an analytic surface S in S_3 by

$$\cdots, U_n, \cdots, U_1, U, V, V_1, \cdots, V_n, \cdots.$$

If the image of one side of the Demoulin quadrilateral associated with S describes in S_5 a net which is conjugate to the corresponding rectilinear congruence $\Gamma_{U_1U_2}$ or $\Gamma_{V_1V_2}$, then the images of the other sides have the same property and S is a projective minimal surface. The converse is true.

In the case where S is projective minimal and has four distinct Demoulin transforms, the above theorems are used to arrive at a harmonic Laplace sequence obtained by Godeaux [Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39** (1953), 156–164; MR **14**, 897]. *A. Schwartz* (New York)

4026:

Su, Buchin. Contributions to the theory of conjugate nets in projective hyperspace. I. Acta Math. Sinica **9** (1959), 446–454. (Chinese. English summary)

Suppose that the associate Laplace sequence of a conjugate net (A_1) in a linear space S_n of dimension $n>4$ be neither periodic nor degenerate, so that a moving frame $\{A_1A_2\cdots A_{n+1}\}$ can be attached to a general point A_1 of the net (A_1) such that $\cdots, (A_5), (A_3), (A_1), (A_2), (A_4), \cdots$ form the Laplace sequence. Let X and Y be two points in the k-dimensional osculating spaces S_k' and S_k'' of the two corresponding net curves at the points A_3 and A_1, respectively. By using E. Cartan's method of exterior differential forms it is proved that if $n\geq 2k\geq 4$, and if the tangent planes of the surfaces (X), (Y) at the points X, Y pass through the points Y, X respectively, then the two points X, Y describe two conjugate nets, each of which is a Laplace transformed net of the other, and the determination of such points X, Y depends on $2k$ arbitrary functions of one argument. This theorem contains as special cases the first two theorems in a paper by the reviewer [Trans. Amer. Math. Soc. **70** (1951), 312–322; MR **13**, 276] and a theorem of P. O. Bell [Proc. Amer. Math. Soc. **3** (1952), 300–302; MR **13**, 775]. The notion of the conjugate, as well as harmonic, relation between a conjugate net and a rectilinear congruence is also generalized in a similar way. *C. C. Hsiung* (Bethlehem, Pa.)

4027:

Švec, Alois. Déformations projectives des systèmes n-conjugués dans S_{2n-1} dont toutes les transformées de Laplace sont dégénérées. Czechoslovak Math. J. **9** (84) (1959), 440–444. (Russian summary)

A manifold V_n described by $x=x(u_1, \cdots, u_n)$ in a projective space S_{2n-1} satisfying a system of equations

$$x_{ij} = 0 \quad (j \neq i;\, j, i = 1, \cdots, n),$$
$$x_{nn} = a(u)x + \sum_{i=1}^{n} b^i(u)x_i + \sum_{\alpha=1}^{n-1} c^\alpha(u)x_{\alpha\alpha}$$

(lower subscript indicating partial derivatives), and admitting a projective deformation of the second order, is such that, under a suitable transformation of the u's, a is a constant k, $b^i = U_i(u_i)$, $c_\alpha = 1$; its projective deformations are obtained by varying k. A recurrent construction of these manifolds is given. *E. Bompiani* (Rome)

4028:

Čech, Eduard. Sur la déformation projective des surfaces développables. Bŭlgar. Akad. Nauk Izv. Mat. Inst. **3**, no. 2, 81–97 (1959). (Bulgarian and Russian summaries)

Two surfaces s, $\bar{s}$ are said to be projectively deformable if a point correspondence X, $\bar{X}$ between the points of s, $\bar{s}$ exists such that for each point X of s a homographic transformation can be found which transports X into the correspondent $\bar{X}$ of $\bar{s}$ and transforms a curve (C) passing through X and belonging to s into a curve (C^*) which has at $\bar{X}$ analytic contact of second order with the curve $(\bar{C})$ which corresponds to (C) on $\bar{s}$. Let s, $\bar{s}$ be two developable surfaces in projective space S_3 and let their generators g, $\bar{g}$ be related to a common parameter t; let φ denote this correspondence between g and $\bar{g}$. Choose for each value of t a projectivity $\pi(t)$ which transforms the points of $g(t)$ into the points of $\bar{g}(t)$. The (∞^1) family of projectivities π constitutes a point transformation P of s into $\bar{s}$. The transformations P are the most general projective deformations of developable surfaces. The subject of the present paper is the study of "special" projective deformations P which are characterized by the property that it is possible to choose the homography $(K=K_0)$ in such a way that the deformation will depend on the generator g but not on the location of X on g. In this case the homography $K_0 = K_0(t)$ is uniquely determined and is called the principal homography. P is always a special deformation if the two surfaces s, $\bar{s}$ are cones; P is never special if only one of the surfaces is a cone; and finally, if s and $\bar{s}$ are formed by tangents to two curves (C), $(\bar{C})$, the projective deformation P can be defined on choosing first the base correspondence φ which is reduced in this case to an arbitrary point correspondence between the two curves (C), $(\bar{C})$ and on choosing next the projectivities π. The special P in this case are characterized by the properties that the two curves (C), $(\bar{C})$ as well as the base correspondence φ can be chosen arbitrarily, whereas the projectivities $\pi(t)$ are subject to the condition that the homographies of S_3 containing π realize analytic contact of at least the first order between (C) and $(\bar{C})$.

Let γ be a curve traced on s, let $\bar{\gamma}$ be the image of γ on $\bar{s}$ (by means of a special deformation P) and let X be a point of the generator $g(t)$ of s and $\bar{X}$ its image by means of P. Consider also the image $K_0\gamma$ by means of the principal homography $K_0(t)$. For any fixed value of t, the two curves $\bar{\gamma}$ and $K_0\gamma$ have at the point $\bar{X} = PX = K_0(X)$ an analytic contact of the second order. The author defines a certain point $B(t)$ with respect to the surfaces s, $\bar{s}$ and the dual point $\bar{B}(t)$ obtained by interchanging the roles of s, $\bar{s}$ in the definition of $B(t)$. He shows that if the point B lies on the tangent to $\bar{\gamma}$ at $\bar{X}$, the two curves $\bar{\gamma}$ and $K_0\gamma$ have at $\bar{X}$ a contact of third order which is analytic if and only if the point B coincides with X. We say that P is of the first kind if B does not lie in the plane Γ tangent to s along g; of the second kind if B lies in Γ; of the third kind if B lies on g and of the fourth kind if B is the point of regression of (C) or the vertex of s if s is a cone.

In the case of the special projective deformations in which s is not a cone there exists for each t a homography H, called the point homography of φ, which realizes an analytic contact of fourth order between the curves (C) and $(\bar{C})$, and dually there exists the plane homography H^* which realizes analytic plane contact of fourth order between the "curves" (C) and $(\bar{C})$ whose elements are the tangent planes of s and $\bar{s}$ at points of (C) and $(\bar{C})$. The kind of deformation P is second or higher if H and H^* transform all the points of g in the same manner. The kind of deformation P is second or higher if, for each t, π is part of H (or of H^*). The kind is third or higher if H and H^* transform in the same manner all the points of the tangent plane Γ and is fourth if H and H^* coincide.
P. O. Bell (Culver City, Calif.)

4029:

Mayer, O. Sur les systèmes triplement conjugués à invariants égaux. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **10** (1959), 277–282. (Romanian. Russian and French summaries)

The systems $x = x(u^1, u^2, u^3)$ in projective P_n satisfy the equations $\partial_{jk}x = a_ix + b_i\partial_jx + c_i\partial_kx$ with the integrability conditions. Here $\partial_jx = \partial x/\partial u^j$, etc., and i, j, k are circular permutations of 1, 2, 3. Added is the condition $\partial_jb_i = \partial_kc_i$ that the Darboux invariants are equal. Integration gives certain expressions for the a, b, c, when the x are conveniently normalized. These expressions are derived for the case of distinct foci, and also for the cases that 1, 2 or 3 pairs of foci coincide. See the author's paper in same Stud. **9** (1958), no. 1, 113–124 [MR **21** #3003].
D. J. Struik (Cambridge, Mass.)

4030:

Dumitraş, Viorel. Sur le groupe de mouvements d'un espace A_n. Com. Acad. R. P. Romîne **9** (1959), 875–878. (Romanian. Russian and French summaries)

If an affine A_n has a stability group $\bar{G}(P)$ which leaves one and only one direction invariant, then the motion group of the space is transitive. The stability group of an A_n which is not locally euclidean cannot have the group of homothetic transformations as a subgroup. [See G. Vranceanu, Rend. Circ. Mat. Palermo (2) **5** (1956), 288–296; MR **19**, 455.] *D. J. Struik* (Cambridge, Mass.)

4031:

Dumitraş, Viorel. À propos des espaces A_3 à groupe intransitif maximum. Com. Acad. R. P. Romîne **9** (1959), 899–903. (Romanian. Russian and French summaries)

The only affine A_3 with an intransitive motion group G_8 are equivalent to spaces of which the connection can be reduced to the form $\Gamma^i_{jk} = \delta_j{}^i\varphi_k + \delta_k{}^i\psi_j$ $(i, j, k = 1, 2, 3)$, $\varphi_1 = \varphi_2 = \psi_1 = \psi_2 = 0$, $\varphi_3 = \Phi(x^3)$, $\psi_3 = \Psi(x_3)$. There are no spaces A_3 with intransitive motion group G_7.
D. J. Struik (Cambridge, Mass.)

4032:

Dumitraş, Viorel. Détermination des espaces A_n à connexion affine non symétrique générale, à groupe

maximum de mouvements. Com. Acad. R. P. Romîne **9** (1959), 1013–1017. (Romanian. Russian and French summaries)

Such spaces are spaces with an absolute parallelism and symmetric in the sense of P. K. Raševskiĭ [Trudy Sem. Vektor. Tenzor. Anal. 8 (1950), 82–92; MR **12**, 534]. They are equivalent to spaces for which the connection coefficients are zero, except $\Gamma^1_{23} = -\Gamma^1_{32} = \frac{1}{2}$. The motion group is transitive with n^2-2n+6 parameters.

D. J. Struik (Cambridge, Mass.)

4033:

Dumitraş, Viorel. Détermination des espaces homogènes A_3 à groupe de mouvements G_5. Acad. R. P. Romîne. Stud. Cerc. Mat. **10** (1959), 331–364. (Romanian. Russian and French summaries)

Study of the A_3 with stability group given by $x^1p_1 - x^2p_2$, x^3p_3, the homogeneous A_3 with x^3p_1, $2x^1p_1+x^2p_2$, those with x^3p_1, x^1p_1, those with x^2p_1, $2x^1p_1+x^2p_2-x^3p_3$, those with x^3p_2, $x^3p_1+x^1p_2$, those with x^3p_1, x^3p_2, and those with a motion group of 5 parameters of which the stability subgroup is given by x^3p_2, x^1p_2.

D. J. Struik (Cambridge, Mass.)

4034:

Dumitraş, Viorel. Groupes de mouvement à 4 paramètres des espaces homogènes A_3. Acad. R. P. Romîne. Stud. Cerc. Mat. **11** (1960), 133–157. (Romanian. Russian and French summaries)

Continuing his classification of affine A_3 with a transitive motion group [same Stud. **6** (1957), 183–234; An. Univ. "C.I. Parhon" Bucureşti. Ser. Şti. Nat. **6** (1957), no. 13, 27–43; MR **20** #2743, 2742], the author shows that there are only 4 linear one-parametric groups which can be full stability groups of a homogeneous A_3. They are $x^1p_1-x^2p_2$, x^1p_1, $x^3p_2+x^2p_1$, x^1p_2. Then the $\Gamma_{bc}{}^a$ are determined for the A_3 with $x^1p_1-x^2p_2$ and x^1p_1 as stability transformations, and similar work is done for homogeneous A_3 with transformations $x^3p_2+x^2p_1$, x^1p_2.

D. J. Struik (Cambridge, Mass.)

4035:

Dumitraş, Viorel. Détermination des espaces A_3 à groupe G_6 intransitif. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **10** (1959), 337–348. (Romanian. Russian and French summaries)

It is shown that there exist two types of A_3 with intransitive G_6. The connection coefficients are determined for the case that the stability group is given by x^3p_1, x^3p_2, x^1p_1, x^2p_2, and for the case x^3p_1, x^3p_2, x^1p_2, $x^1p_1+x^2p_2$. There are no such A_3 with a one-dimensional manifold of intransitivity, and there are no such A_3 without torsion.

D. J. Struik (Cambridge, Mass.)

4036:

Postelnicu, Tiberiu. Linear connections associated with the projective plane. Rev. Math. Pures Appl. **4** (1959), 115–122.

Making use of results of his previous paper, Com. Acad. R. P. Romîne 8 (1958), 13–18 [MR **21** #2270], and also of results of Vranceanu [Boll. Un. Mat. Ital. (3) **12** (1957), 145–153; MR **19**, 764], the author obtains in explicit form a complete catalogue of the essentially different linear connections associated with the projective plane.

T. J. Willmore (Liverpool)

4037:

Fukami, Tetsuzo. Affine connections in almost product manifolds with some structures. Tôhoku Math. J. (2) **11** (1959), 430–446.

This paper is concerned with the study of affine connexions in a manifold with a system of distributions. The author states that A. G. Walker [Quart. J. Math. Oxford Ser. (2) **6** (1955), 301–308; **9** (1958), 221–231; MR **19**, 312; **20** #6135] and also the present reviewer [ibid. **7** (1956), 267–276; MR **20** #4299] obtained an affine connexion in the large with respect to which the distributions are parallel and which is symmetric when the system is integrable. He further states that the object of his paper is to give a general method for obtaining all affine connexions with respect to which the distributions are parallel. It appears that he does not realize that Walker did precisely this in the two papers mentioned above, and since essentially the same methods are used as in Walker's papers, it is difficult to see the point of this part of the paper. The last part of the paper considers affine connexions specially related to manifolds which admit almost complex, quaternionic or Hermitian structures.

T. J. Willmore (Liverpool)

4038:

Ahn, Jae Koo. On the parameter groups manifolds. Kyungpook Math. J. **2** (1959), 39–45.

A study is made on the parameter group manifold associated with an r-dimensional continuous transformation group with the operation of the extended exterior differentiation defined by H. Flanders [Trans. Amer. Math. Soc. **75** (1953), 311–326; MR **15**, 161] as an affine connection in the sense of N. Horie [Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **30** (1956), 23–42; MR **19**, 753]. The curvature and torsion forms of the group manifold are computed, and some identities concerning these forms are obtained.

C. C. Hsiung (Bethlehem, Pa.)

4039:

Кон-Фоссен, С. Э. [Cohn-Vossen, S. E.] ★Некоторые вопросы дифференциальной геометрии в целом. [Some problems of differential geometry in the large.] Edited by N. V. Efimov. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1959. 303 pp. 10.35 r.

A collection, prefaced by a brief survey by N. V. Efimov, of the following articles by Cohn-Vossen, translated into Russian: "Bending of surfaces in the large", Uspehi Mat. Nauk **1** (1936), 33–76; "Unstarre geschlossene Flächen", Math. Ann. **102** (1929), 10–29 [also Uspehi Mat. Nauk **9** (1954), no. 1 (59), 63–81; MR **15**, 819]; "Singularitäten konvexer Flächen", Math. Ann. **97** (1927), 377–386; "Die parabolische Kurve", ibid. **99** (1928), 273–308; "Kürzeste Wege und Totalkrümmung auf Flächen", Compositio Math. **2** (1935), 69–133; "Totalkrümmung und geodätische Linien auf einfachzusammenhängenden offenen vollständigen Flächenstücken", Mat. Sb. **1 (43)** (1936), 139–164; "Existenz kürzester Wege", Compositio Math. **3** (1936), 441–452.

4040:

Pogorelov, A. V. A theorem on infinitesimal bending of general convex surfaces. Dokl. Akad. Nauk SSSR **127** (1959), 969–970. (Russian)

This paper precedes the author's article in same Dokl.

128 (1959), 475–477 [MR **21** #7545] which comprises the present results. *H. Busemann* (Los Angeles, Calif.)

4041:

Stoka, Marius I. Sulla misura cinematica in uno spazio euclideo E_n. Boll. Un. Mat. Ital. (3) **14** (1959), 467–476. (English summary)

After some general remarks about the kinematic measure in E_n, the author observes that the measure of sets of circles, hyperbolas and spheres coincides with the kinematic measure of certain particular groups and gives the explicit expression for these measures.
L. A. Santaló (Buenos Aires)

4042:

Stoka, Marius I. The measure of a set of manifolds in the space E_3. Rev. Math. Pures Appl. **4** (1959), 305–316. (Russian)

Russian version of the author's previous paper in Acad. R. P. Romîne. Stud. Cerc. Mat. **9** (1958), 547–558 [MR **21** #2285]. *J. G. Wendel* (Ann Arbor, Mich.)

GENERAL TOPOLOGY, POINT SET THEORY

See also 3705, 3967.

4043:

Császár, Ákos. ★Fondements de la topologie générale. Akadémiai Kiadó, Budapest, 1960. 231 pp. $6.00.

This monograph works out in detail the author's generalization of topological, uniform, and proximity spaces [Rev. Math. Pures Appl. **2** (1957), 399–407; MR **20** #1289 and erratum]. The basic concept is renamed a "syntopogenic" space. It consists of a set E and a family S of quasi-orderings of 2^E, each satisfying (i) $0<0$, $E<E$, (ii) if $A<B$ then $A\subset B$, (iii) if $A\subset A'<B'\subset B$ then $A<B$, and (iv) if $A<B$ and $A'<B'$ then $A\cap A'<B\cap B'$ and $A\cup A'<B\cup B'$; moreover, any two orderings in S have a common refinement in S, and for any ordering $<$ in S there is $<'$ in S such that whenever $A<B$ there is C such that $A<'C<'B$.

There are three additional conditions which taken in pairs determine respectively the topological, uniform, and proximity spaces: (a) S consists of a single ordering; (b) (iv) is true for infinite unions; (c) for each $<$ in S, $A<B\Rightarrow E-B<E-A$. (All three together determine the discrete spaces.) Because of the symmetry (in (c) and elsewhere) these are essentially the usual uniform and proximity spaces; if they are T_0, they are completely regular. However, all topological spaces are included.

The first ten chapters develop the elementary meanings of these conditions, the admissible ("continuous") mappings of syntopogenic spaces, and the connections with classical topology, uniformity, proximity. Chapter 11 describes direct products, defined by the weakest syntopogenic structure making projections continuous. Apparently none of (a), (b), (c) is preserved under formation of products, but the usual products of classical spaces are "associated" in a canonical manner with the syntopogenic products.

Chapter 12 presents the surprising result (credited to J. Czipszer) that every syntopogenic structure is equivalent to the weak structure induced by a family of real-valued functions. For this, of course, the real line must be given a "one-sided" structure in which the closed sets are the intervals $[a,\infty)$, with corresponding quasi-uniform and -proximity structure.

Chapter 13 relates quasi-uniform structures to families of non-symmetric distance functions; Chapter 14 treats separation axioms; and the concluding Chapters 15–16 treat convergence, completion and compactification. Every syntopogenic space can be embedded in a complete space (a topogenic space, i.e., one satisfying (a), is already complete according to the present definition; an unfortunate side effect is that not every convergent filter is Cauchy); every topogenic space can be embedded in a compact topogenic space. *J. Isbell* (Seattle, Wash.)

4044:

Herzog, Fritz; Kelly, L. M. A generalization of the theorem of Sylvester. Proc. Amer. Math. Soc. **11** (1960), 327–331.

The following generalization of a theorem of Sylvester is established: If a finite collection of disjoint compact sets in E_n, at least one of which is infinite, is such that every straight line intersecting two distinct sets intersects at least one additional set, then all the sets are subsets of the same line. Examples are presented to show the necessity of the assumptions. (E.g., the nine points of the Pappus configuration may be arranged in three groups in such a way that a line cutting any two of them intersects the third.) {Another generalization of Sylvester's theorem, weaker than the present one for $n=2$, was given by Edelstein [Riveon Lematematika **11** (1957), 50–55; MR **21** #2211].} *B. Grünbaum* (Seattle, Wash.)

4045:

Iséki, Kiyoshi. On locally bounded functions. Proc. Japan Acad. **35** (1959), 279–280.

4046:

Banaschewski, Bernhard. On homeomorphisms between extension spaces. Canad. J. Math. **12** (1960), 252–262.

The Stone-Čech compactification of a given completely regular space, Katětov's maximal Hausdorff-closed extension of a Hausdorff space, Hewitt's "Q-extension" of a completely regular space, etc., are examples of methods that assign to each space X of a class Γ a space $\gamma X\in\Gamma$ which contains X densely. This process is here treated axiomatically with attention given to the following question: what conditions on $\{\Gamma,\gamma\}$ ensure that if γX and γY are homeomorphic, then X and Y are homeomorphic?
R. Arens (Los Angeles, Calif.)

4047:

Leader, S. On completion of proximity spaces by local clusters. Fund. Math. **48** (1959/60), 201–216.

The author gives a new method of defining Yu. M. Smirnov's completion of a proximity space [Dokl. Akad. Nauk SSSR **88** (1953), 761–764; **91** (1953), 1281–1284; MR **15**, 144; **16**, 58], making use of pseudo-metrics and also the ideas of his previous paper, Fund. Math. **47** (1959), 205–213 [MR **22** #2978]. He also discusses, with

respect to these notions, several properties of mappings on a proximity space to another and convergence of mappings on an abstract set into a proximity space. Concerning these discussions, two conjectures are given.

{Reviewer's remark: Conjecture 2 is true, since the notion of uniform convergence of mappings into a uniform space determines its uniformity.}

I. G. Amemiya (Tokyo)

4048:

Brown, Thomas A.; Comfort, W. W. New method for expansion and contraction maps in uniform spaces. Proc. Amer. Math. Soc. **11** (1960), 483–486.

Freudenthal and Hurewicz showed that if the function f, from the totally bounded metric space M onto M, has the property that $(fx, fy) \leq (x, y)$ for each x and y in M, then f is an isometry. Rhodes proved that an even stronger result holds in the more general setting of uniform spaces. The present paper offers a theorem similar to that of Rhodes, together with a number of results concerning "expansion" maps in uniform spaces.

A. Edrei (Syracuse, N.Y.)

4049:

Obrechkoff, N. Sur la convergence du procédé itératif dans l'analyse. II. C. R. Acad. Bulgare Sci. **12** (1959), 395–398. (Russian summary)

[For part I see same C. R. **12** (1959), 97–99; MR **21** #4210.] Theorem: Let f map the connected domain D of the complex plane into itself. Suppose that $|f(z)-f(u)| < |z-u|$ for all $z, u \in D$, $z \neq u$. If the equation $f(z)=z$ has a solution $\zeta \in D$, then the solution is unique. Moreover, for any $\alpha \in D$ the sequence α_n converges to ζ. Here $\alpha_1 = \alpha$, $\alpha_{n+1} = f(\alpha_n)$. The author also gives conditions on f and D which insure the existence of such a ζ. The theorem is extended to an arbitrary metric space.

R. R. Goldberg (Evanston, Ill.)

4050:

Hadwiger, H. Ein Satz über stetige Funktionen auf der Kugelfläche. Arch. Math. **11** (1960), 65–68.

A simple elementary proof is given that for two continuous functions f, g defined on a 2-dimensional sphere S^2 points p, q exist on S^2 at a given spherical distance σ, $0 \leq \sigma \leq \pi$, apart so that $f(p)=f(q)$ and $g(p)=g(p^*)$, $g(q)=g(q^*)$, where p^*, q^* are antipodal points of p, q. If σ is specialized to π this becomes an earlier result due to Borsuk and Ulam; if f and g are identical it becomes Livesay's generalization [Ann. of Math. (2) **59** (1954), 227–229; MR **15**, 548] of a result of Dyson [ibid. **54** (1951), 534–536; MR **13**, 450].

D. Derry (Vancouver, B.C.)

4051:

Weier, Joseph. Ueber Transformation von Kompakten in die Sphäre. Proc. Japan Acad. **35** (1959), 599–602.

The author proves: (a) If $f, g: P \to Q$, where P, Q are finite polytopes, and f is essential, $g \simeq 0$, then $f(p)=g(p)$ for some $p \in P$; (b) same conclusion if P is compact metric, Q a closed Euclidean manifold.

J. Dugundji (Los Angeles, Calif.)

4052:

Nagami, Keiô. Finite-to-one closed mappings and dimension. II. Proc. Japan Acad. **35** (1959), 437–439.

[For part I see same Proc. **34** (1958), 503–506; MR **21** #862.] Theorem 1 states that, given any metric space R_0 of finite positive dimension n, and any non-negative integer $m < n$, there exists a closed (continuous) mapping π_0 of some m-dimensional metric space T onto R_0 such that, for each $y \in R_0$, $\pi_0^{-1}(y)$ consists of at most $n-m+1$ points. This answers a question of Hurewicz [answered, when R_0 is separable, by J. H. Roberts, Duke Math. J. **8** (1941), 565–574; MR **3**, 138]. The proof is sketched, and some further theorems are announced without proof: for example, (theorem 5) if f is a closed mapping of a metric space R onto a (necessarily metric) space S such that, for each $y \in S$, the boundary of $f^{-1}(y)$ is finite, and if R is the union of countably many 0-dimensional spaces, then so is S.

A. H. Stone (Manchester)

4053:

Levšenko, B. T. Strongly infinite-dimensional spaces. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1959**, no. 5, 219–228. (Russian)

This paper organizes several natural notions of strong infinite-dimensionality, which turn out to be mutually equivalent for countably compact normal spaces. As the author observes, his results and those of E. Sklyarenko [Izv. Akad. Nauk SSSR Ser. Mat. **23** (1959), 197–212; MR **21** #5179] support the following defining condition: There exists a sequence of disjoint pairs of closed sets (A_n, B_n) such that any family of closed sets C_n each separating A_n and B_n has the finite intersection property. This is Yu. Smirnov's modification of P. S. Aleksandrov's definition, which requires non-empty total intersection. The other conditions (four of them) require mappings into a product of $\aleph_0$ lines or intervals which are in some sense essential. For example, a normal space is strongly infinite-dimensional in Smirnov's sense if and only if it admits a sequence of mappings f_i into $[0, 1]$, the first n of which, for each n, define an essential mapping upon the n-cube. {Reviewer's remark: For non-normal spaces this seems the more natural defining condition.}

It is shown (theorem 4) that a metrizable space which is strongly infinite-dimensional in the sense of Aleksandrov cannot be a countable union of subspaces which fail to have this property. Nor (theorem 5) can it be a product of a zero-dimensional space and a compact space which is not strongly infinite-dimensional. For normal spaces (theorem 2), X is strongly infinite-dimensional in the sense of Smirnov if and only if βX is strongly infinite-dimensional. A compact strongly infinite-dimensional space (theorem 6) must contain a closed subspace which is not separated by any of its finite-dimensional subspaces. Since Sklyarenko has since improved theorem 6 [op. cit.], the proof is merely sketched.

J. Isbell (Seattle, Wash.)

4054:

Mardešić, Sibe. Infinite Cartesian products and a problem concerning homology local connectedness. Trans. Amer. Math. Soc. **93** (1959), 395–417.

The author gives a category of 2-dimensional compacta which are lc^1 (singular homology) but not LC^1. (Infinite-dimensional such examples are easily constructed.) This is a consequence of the general result: Let K be any finite cell-complex with a single vertex and at least one 1-cell. Then in the infinite cartesian product $\prod K = K_\infty$

there are compact connected metrizable LC^0 subspaces A, B, with $\dim A=1$, $\dim B=2$, $A\subset B$, such that (1) the injection $\pi_1(B)\to\pi_1(K_\infty)$ is an isomorphism onto; (2) in the space B, every point of $B-A$ has neighborhoods homeomorphic to E^2 and every point of A has a basis of connected neighborhoods $\{U\}$ such that each injection $\pi_1(U)\to\pi_1(B)$ is a monomorphism with image isomorphic to $\pi_1(K_\infty)$. The examples desired follow by taking K such that $\pi_1(K)$ is a non-trivial perfect finite group G, and showing $\prod G$ (direct product) is not trivial for such groups. As a further example, polytopes P_n $(n=1, 2, 3, \cdots)$ with $H_i(P_n)=0$ for all $i, n\geq 1$ but with $H_1(\prod P_n)\neq 0$ are exhibited. *J. Dugundji* (Los Angeles, Calif.)

4055:

Dugundji, J. Absolute neighborhood retracts and local connectedness in arbitrary metric spaces. Compositio Math. **13**, 229–246 (1958).

This paper extends the theory of LC^n and ANR spaces, standard for separable metric spaces, to arbitrary metric spaces. Most of the results were obtained independently by Y. Kodama [Proc. Japan Acad. **33** (1957), 79–83; MR **19**, 671]. In particular, for finite-dimensional metric spaces Y, the following are equivalent: (1) Y is LC^n, where $n=\dim Y$; (2) Y is LC^∞; (3) Y is locally contractible; (4) Y is an ANR (for metric spaces). If these conditions hold, each of the following is necessary and sufficient for Y to be an AR: (1) Y is C^n; (2) Y is C^∞; (3) Y is contractible. Infinite-dimensional spaces are also considered, and characterizations of LC^n and ANR in terms of partial realization [cf. Lefschetz, *Topics in topology*, Princeton Univ. Press, Princeton, N.J., 1942; MR **4**, 86] and "factorization" (domination by a CW polyhedron) are given. Some of the results are shown to apply also to (not necessarily metric) CW polyhedra. The proofs are decidedly more complicated than in the separable case; for example, one imbeds Y suitably in a Banach space instead of in Euclidean or Hilbert space. {As E. Michael has observed to the reviewer, some of the details could have been simplified by choosing metrics for Y in which the local properties hold uniformly locally.}
A. H. Stone (Manchester)

ALGEBRAIC TOPOLOGY

See also 3751, 3752, 4054, 4055, 4071.

4056:

Aleksandrov, P. S. ★Combinatorial topology. Vol. 3. Graylock Press, Albany, N.Y., 1960. viii+148 pp. $6.50.

The final volume of the translation by H. Komm, constituting Parts 4 and 5 of the original [OGIZ, Moscow-Leningrad, 1947; MR **10**, 55]; translations of Vols. 1 and 2 are listed in MR **17**, 882 and **19**, 759.

4057:

★Topologie algébrique et géométrie différentielle. Colloques Internationaux du Centre National de la Recherche Scientifique, LXXXIX, Lille, 1–6 juin 1959. Éditions du C.N.R.S., Paris, 1960. 189 pp.

Also published as Bull. Soc. Math. France **87** (1959), fasc. 4. Articles by J. F. Adams, M. Berger, R. Bott, E. Calabi and E. Vesentini, J. Cerf, A. Dold, H. Grauert, A. Haefliger, G. Hirsch, M. F. Atiyah and F. Hirzebruch, M. A. Kervaire, J.-L. Koszul, P. Libermann, A. Lichnerowicz, J. W. Milnor, G. Reeb, W. Shih, R. Thom; to be reviewed individually.

4058:

Sasao, Seiya. On a certain cup product. J. Math. Soc. Japan **11** (1959), 112–115.

In most cases arguments involving the Steenrod functional cup-product associated with a map $f: X\to Y$ can be replaced by arguments in terms of the cohomology of the space obtained from Y by attaching a cone on X according to f. In the reviewer's article, Proc. Amer. Math. Soc. **8** (1957), 374–383 [MR **19**, 974], the latter type of argument is used to establish a relation between cohomology and the relative Whitehead product of Blakers and Massey. The author gives an alternative proof of this relation by means of functional cup products. *I. M. James* (Oxford)

4059:

Chow, Sho-Kuan. The relations between homotopy groups and homology groups, and some of its applications. Acta Math. Sinica **7** (1957), 346–369. (Chinese. English summary)

#4060 below is a translation.

4060:

Chow, Sho-kwan. The relations between homotopy groups and homology groups and some of their applications. Sci. Sinica **7** (1958), 686–703.

If Q is any non-empty set of prime numbers, let $C(Q)$ denote the class of all abelian groups A which are torsion groups and have the property that the p-primary component of A is 0 for p not in Q; if Q is empty, let $C(Q)$ denote the class of abelian groups having only a zero element. For any positive integer k, let $S(k)$ denote the set of all primes p such that $2p-3\leq k$. With these conventions, the author's main theorem may be stated as follows. Let Q be any set of primes and let X be a simply connected space such that $\pi_i(X)\in C(Q)$ for $i<n$. Then the Hurewicz homomorphism $\pi_m(X)\to H_m(X)$ is a $C(Q\cup S(m-n))$-monomorphism for $n\leq m\leq 2n-2$ and is a $C(Q\cup S(m-n-1))$-epimorphism for $n\leq m\leq 2n-1$. Here the terms "$C(Q)$-monomorphism" and "$C(Q)$-epimorphism" are used in the sense of Serre's fundamental paper [Ann. of Math. (2) **58** (1953), 258–294; MR **15**, 548].

The author also proves theorems of this same general type about relative and triad homotopy groups.
W. S. Massey (New Haven, Conn.)

4061:

Chang, Su-cheng. On invariants of homotopy groups of spheres. Sci. Record (N.S.) **4** (1960), 88–90.

This brief note is a sequel to Acta Math. Sinica **4** (1954), 201–221 [MR **17**, 290]. The reduced product construction is used to define a family of homomorphisms which may be useful for analysing homotopy groups of spheres. The simplest of these is a homomorphism $H_{p-1}: \pi_{i+1}(S^{n+1})\to\pi_i(S^{pn})$ $(p=1, 2, \cdots)$ defined for $i<(p+1)n$; the second

maps the kernel of this into a certain factor group; and so forth. Some results are proved about them. [Cf. #4062 below.]
I. M. James (Oxford)

4062:

Chang, Su-cheng. On invariants associated with homotopy groups of spheres. Acta Math. Sinica **9** (1959), 468–474. (Chinese. English summary)

By means of the reduced products, the author introduces in this paper a sequence of $2(p-1)$ homomorphisms H_{p-1}, K_{p-1}, H_{p-2}, K_{p-2}, $\cdots$, H_1, K_1, where H_{p-1} is defined on the homotopy group $\pi_{pn+r+1}(S^{n+1})$ of the $(n+1)$-dimensional sphere S^{n+1} $(0 \leq r \leq n-1)$, K_j is defined on $H_j^{-1}(0)$ and H_j is defined on $K_{j+1}^{-1}(0)$. The main theorem in this paper states that, to each element α of $\pi_{pn+r+1}(S^{n+1})$, either there is an element β of $\pi_{pn+r}(S^n)$ such that $\alpha = E\beta$, E being the suspension homomorphism, or there is one of the $2(p-1)$ homomorphisms mentioned above, denoted by G, such that G is defined on α and $G(\alpha) \neq 0$.
S. T. Hu (Los Angeles, Calif.)

4063:

Poenaru, Valentin. Über eine Klasse von dreidimensionalen Mannigfaltigkeiten. I. Rev. Math. Pures Appl. **4** (1959), 651–659.

Let T^3 be the solid torus bounded by S^2: $(x = (R + r\cos\theta) \cdot \cos\psi,\ y = (R + r\cos\theta)\sin\psi,\ z = r\sin\theta)$. Let $\hat{\alpha}$ and $\hat{\beta}$ be the generators of $H_1(S^2)$ represented by $\psi = 0$ and $\theta = 0$. Another pair of generators is $\hat{x} = a\hat{\alpha} + b\hat{\beta}$, $\hat{y} = c\hat{\alpha} + d\hat{\beta}$ for each unimodular matrix $\mu = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let $x \in \hat{x}$, $x' \in \hat{x}$, $y \in \hat{y}$ be such that $x \cap y$ is a point, $x' \cap y$ is a point, and $x \cap x' = \emptyset$. A manifold $V^3(\mu)$ is obtained from T^3 by identifying the two bands into which $x \cup x'$ separates S^2 under a homeomorphism $S^2 \to S^2$ which leaves $x \cup x'$ pointwise fixed and maps onto one another the two arcs into which $x \cup x'$ divides y. It is shown that if $V_3(\mu)$ is simply connected, it is a 3-sphere; then, that the same conclusion follows from $H_1(V_3(\mu)) = 0$. The general $V^3(\mu)$ is proved to be obtainable from a 3-sphere by (1) deleting from the latter a set of regions bounded by tori with certain linking properties, then (2) identifying such tori in pairs according to a specified scheme.
S. S. Cairns (Urbana, Ill.)

4064:

Hu, Sze-Tsen. Isotopy invariants of topological spaces. Proc. Roy. Soc. London. Ser. A **255** (1960), 331–366.

In this paper, the author uses the following definitions. A continuous map $f: X \to Y$ is called an imbedding if it carries the space X homeomorphically onto the subspace $f(X)$ of Y. Two spaces X and Y are of the same isotopy type if there exist imbeddings $f: X \to Y$ and $g: Y \to X$ such that the composed imbeddings fg and gf are isotopic to the identity maps. The author is concerned with the problem of finding invariants of the isotopy type of a space. He proposes essentially two such invariants for consideration, as follows. Let X^n denote the n-fold cartesian product of X with itself, and consider X imbedded in X^n by the diagonal map. Define $R_n(X) = X^n - X$ and define $E_n(X)$ to be the space of all paths $\sigma: [0, 1] \to X^n$ in X^n such that $\sigma(t) \in X$ if and only if $t = 0$. Then the homotopy types of the spaces $R_n(X)$ and $E_n(X)$ are isotopy invariants of X. In particular, the homology and cohomology groups of $R_n(X)$ and $E_n(X)$ are isotopy invariants of X. In case X is a finite simplicial complex, the author proves that the spaces $R_n(X)$ and $E_n(X)$ are of the same homotopy type as certain well defined subcomplexes of the product complexes X^n and X^{n+1} respectively. The proof of this fact is based on a certain cellular decomposition of the pair (X^n, X) given by W.-T. Wu [Acta Math. Sinica **3** (1953), 261–290; MR **17**, 290]. Thus in principle these isotopy invariants are computable in case X is a finite simplicial complex.

In the last part of the paper the author gives some examples of pairs of graphs (connected 1-dimensional complexes) which have the same homotopy type and are such that for each integer $k \neq 2$ both have the same number of vertices of order k, but are not of the same isotopy type. This last assertion is proved by computing the homology of the spaces $R_2(X)$ for each graph X of the pair.
W. S. Massey (New Haven, Conn.)

4065:

Wu, Cheng-der. On the modulo 2 imbedding class of triangulable compact manifolds. Acta Math. Sinica **10** (1960), 22–32. (Chinese. English summary)

In this paper, the author proves the following theorem: Let M^n be an n-dimensional $(n > 0)$ triangulable compact manifold. Then $\Phi^N(M^n) \equiv 0 \pmod 2$ if and only if $S_m{}^k H^r(M^n; I_2) = 0$ whenever $2k + r \geq N$. Together with known results, this theorem implies that the following three statements are equivalent for any n-dimensional $(n > 0)$ triangulable compact differentiable manifold M^n: (i) $\Phi^N(M^n) \equiv 0 \pmod 2$; (ii) $\overline{W}^k(M^n) = 0$ whenever $k \geq N - n$; (iii) $S_m{}^k H^r(M^n; I_2) = 0$ whenever $2k + r \geq N$.
S. T. Hu (Los Angeles, Calif.)

DIFFERENTIAL TOPOLOGY

See also 4057.

4066:

Kneser, Hellmuth; Kneser, Martin. Reell-analytische Strukturen der Alexandroff-Halbgeraden und der Alexandroff-Geraden. Arch. Math. **11** (1960), 104–106.

The Alexandroff half-line A^+ is defined to be the set of all pairs $(\mu, \xi) \neq (0, 0)$ with μ a countable ordinal and ξ a real number in $[0, 1)$, the topology in A^+ being the order topology given by the lexicographical ordering of A^+. In an earlier work [Ann. Acad. Sci. Fenn. A I No. 251/5 (1958); MR **21** #133] H. Kneser showed the existence of a real analytic structure for A^+. In the present note the authors show that this structure is not unique. In addition they show that any two segments $\{x: x > z\}$ and $\{x: x > w\}$ are not analytically homeomorphic, and that each analytic mapping of A^+ into itself is either constant or the identity.
H. L. Royden (Stanford, Calif.)

4067:

Royden, H. L. The analytic approximation of differentiable mappings. Math. Ann. **139**, 171–179 (1960).

Theorems are proved concerning the possibility of approximating differentiable maps of closed analytic manifolds by analytic ones. The main tool is the theorem of Morrey that a closed analytic manifold admits an

analytic imbedding in Euclidean space. The author proves that if M and N are closed analytic manifolds then a continuous map from M to N is homotopic to an analytic one. If two analytic maps from M to N are homotopic, then there is a homotopy between them which is analytic on $M \times [0, 1]$. Also regular homotopies of analytic manifolds correspond in the same way to regular homotopies which are analytic.

S. Smale (Berkeley, Calif.)

4068:

Kervaire, Michel A. Sur l'invariant de Smale d'un plongement. Comment. Math. Helv. **34** (1960), 127–139.

To each immersion $f: S^p \to R^{p+q}$ of a sphere in Euclidean space, one can associate the Smale invariant, an element c_f of $\pi_p(V_{p+q,p})$ where $V_{p+q,p}$ is the Stiefel manifold of p-frames in R^{p+q}. The reviewer has proved that two immersions are regularly homotopic if and only if they have the same Smale invariant. The main theorem of the author is that for an imbedding $f: S^p \to R^{p+q}$, with $p \leq 2q-2$, the Smale invariant vanishes and hence f is regularly homotopic to the standard sphere. Subsequently Haefliger and Hirsch have given a proof of this fact. The author ends the paper by discussing the still unsettled case $f: S^p \to R^{p+1}$. He proves in collaboration with Milnor that if f is an imbedding, $c_f = 0$ when $p \not\equiv 0, 1 \bmod 8$. (Apparently the author means to restrict his imbedding to have normal degree 1, for otherwise this is not true.)

S. Smale (Berkeley, Calif.)

4069:

Morse, Marston. Fields of geodesics issuing from a point. Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 105–111.

The goal of this note is to prove a technical theorem to be used in the author's paper discussed in the following review. The main idea is to introduce coordinates with useful properties in a neighborhood of a regular arc in a Riemannian manifold. *S. Smale* (Berkeley, Calif.)

4070:

Morse, Marston. The existence of polar non-degenerate functions on differentiable manifolds. Ann. of Math. (2) **71** (1960), 352–383.

A polar function on a C^∞ closed connected n-manifold is a C^∞ non-degenerate function with just one critical point of index zero and just one critical point of index n. The main object of this paper is to prove that on every closed connected C^∞ manifold there exist polar functions. By an earlier result of the author there exist C^∞ non-degenerate functions on the manifold. It is by modification of one of these functions that the polar function is obtained. The polar function will have the same type numbers as the original function except in dimensions $0, 1, n-1$ and n. Recently the reviewer has obtained generalizations of this theorem [Bull. Amer. Math. Soc. **66** (1960), 373–375; Ann. of Math., to be published].

S. Smale (Berkeley, Calif.)

4071:

Morse, Marston. Topologically non-degenerate functions on a compact n-manifold M. J. Analyse Math. **7** (1959), 189–208.

The author develops the theory of a non-degenerate function on a topological manifold analogous to his now classical theory for the differentiable case. A T-ordinary point of a real function F on a topological manifold M is a point where locally F is topologically equivalent to a regular point of a differentiable function. A T-critical point of F of T-index r is defined in the same way using a non-degenerate critical point of index r of a differentiable function. Then a T-non-degenerate function on a topological manifold is a function with only T-ordinary points and T-critical points of index r, for various r. However in contrast to the differentiable case the author does not know whether on a topological manifold there exist any T-non-degenerate functions. He then proves that the Morse relations are valid in this new setting. The paper ends with application in dimensions 2 and 3.

S. Smale (Berkeley, Calif.)

4072:

Al'ber, S. I. On periodicity problems in the calculus of variations in the large. Amer. Math. Soc. Transl. (2) **14** (1960), 107–172.

Translation of Uspehi Mat. Nauk **12** (1957), no. 4 (76), 57–124 [MR **19**, 751].

4073:

Dynkin, E. B. Homologies of compact Lie groups. Amer. Math. Soc. Transl. (2) **12** (1959), 251–300.

For a review of the Russian original [Uspehi Mat. Nauk (N.S.) 8 (1953), no. 5 (57), 73–120; **9** (1954), no. 2 (60), 233], see MR **15**, 601; **16**, 334.

4074:

Dynkin, E. B. Topological characteristics of homomorphisms of compact Lie groups. Amer. Math. Soc. Transl. (2) **12** (1959), 301–342.

For a review of the Russian original [Mat. Sb. (N.S.) **35** (77) (1954), 129–173], see MR **16**, 673.

4075:

Kervaire, Michel A. Some nonstable homotopy groups of Lie groups. Illinois J. Math. 4 (1960), 161–169.

The author computes some of the non-stable homotopy of $SO(m)$ and $U(m)$, and applies the results to the fibering $SO(m)/SO(m-1) = S^{m-1}$. He calculates the following: (a) $\pi_{r+2m}(U(m))$ for all m, and $r=1, 2$; for $r=1$, these groups depend only on the parity of m. (b) $\pi_{r+m}(SO(m))$ for all $m \geq 8$ and $-1 \leq r \leq 4$; for each r, these groups depend only on the residue of m mod 8 (with the three possible exceptions: $r=2, 3, 4$ and $m \equiv 6-r \bmod 8$; in each case, it is determined that $\pi_{r+m}(SO(m))$ is one of two possible groups, but it is unsettled as to which alternative obtains, and whether or no this alternative depends on the actual value of m rather than on its residue mod 8). The computations rely heavily on Paechter's calculation of $\pi_r(V_{n,m})$. A variety of results follow by his applying his results to the above fibering. For example, if $\partial: \pi_m(S^{m-1}) \to \pi_{m-1}(SO(m-1))$ is the boundary homomorphism, and η_{m-1} the generator of $\pi_m(S^{m-1})$, then $\partial\eta_2 = \partial\eta_6 = 0$, $\partial\eta_{4n-2} \neq 0$, $n \geq 3$; with the Whitehead-Hilton results, then, $\partial\eta_k = 0$, $k \geq 10$, if and only if $k \equiv 3 \bmod 4$.

J. Dugundji (Los Angeles, Calif.)

4076:

Kashiwabara, Shōbin. The structure of a Riemannian manifold admitting a parallel field of one-dimensional tangent vector subspaces. Tôhoku Math. J. (2) **11** (1959), 327–350.

It is a consequence of a theorem of G. de Rham [Comment. Math. Helv. **26** (1952), 328–344; MR **14**, 584] that a complete, simply connected, n-dimensional Riemannian manifold M which admits a parallel field of 1-dimensional tangent vector subspaces is isometric to the product of Riemannian $(n-1)$- and 1-dimensional manifolds. In this paper the author studies the global structure of a manifold M satisfying all of the above conditions except the simple connectedness. Complementary to the 1-dimensional spaces one obtains a parallel field of $(n-1)$-dimensional subspaces. Both fields are completely integrable and lead to integral curves (S-subspaces), which are in fact geodesics, in the first case and integral $(n-1)$-manifolds (R-subspaces) in the second case. Subject to certain conditions, e.g., that the R-subspaces actually are subspaces in the topological sense, or, e.g., that the fundamental group is cyclic, the author gives the structure of M, showing it to be isometric to one of six possible types of manifolds obtained (by suitable identification of points) from the cartesian product of an arbitrary $(n-1)$-manifold with a line, ray, or line segment. Some information is also given in the general case, that is, for M satisfying only the original hypotheses.

W. M. Boothby (St. Louis, Mo.)

4077:

Look, K. H. An analytic invariant and its characteristic properties. Sci. Record (N.S.) **1** (1957), 307–310.

Der Verfasser versucht mit Hilfe von Strukturinvarianten die analytische Isomorphie von Gebieten $D\subset\mathfrak{C}^n$ zu entscheiden. Im Falle, daß D ein irreduzibles klassisches (homogenes) Gebiet ist, gelingt es ihm, zwei Invarianten $L(D)$ und $k_0(D)$ zu finden, die D charakterisieren. Er findet dabei zwei Gebiete R_{III} und R_{IV} der Cartanschen Klassifizierung, die zueinander isomorph sind. Bislang hatte man, wie der Verfasser angibt, angenommen, daß R_{III} und R_{IV} analytisch inäquivalent sind.

H. Grauert (Göttingen)

4078:

Ramspott, Karl-Josef. Existenz von Holomorphiegebieten zu vorgegebener erster Bettischer Gruppe. Math. Ann. **138** (1959), 342–355.

It is well known [Serre, Colloque sur les fonctions de plusieurs variables (Bruxelles, 1953), pp. 57–68, Thone, Liège, 1953; MR **16**, 235] that the Betti groups $B_k(M^n)$ of an n-dimensional holomorphically complete manifold M^n vanish provided $k>n$. The author proves that for any given countable, torsion-free abelian group G there is a domain D of holomorphy in C^2 whose first Betti group $B_1(D)$ is isomorphic to G. For that purpose he exhausts G by certain subgroups G_n $(n=1, 2, \cdots)$, and constructs for each subgroup G_n a domain of holomorphy D_n fulfilling $B_1(D_n)=G_n$, in such a way that these domains D_n converge; the limit domain D turns out to fulfill the above-mentioned requirements.

H. Roehrl (Minneapolis, Minn.)

4079:

Holmann, Harald. Zur Abbildungstheorie komplexer Mannigfaltigkeiten. Math. Ann. **138** (1959), 428–441.

Let $p_1, \cdots, p_n$ be integers whose greatest common divisor is 1. An open subset U of the complex number space C^n is called a $(p_1, \cdots, p_n)$-domain if U is invariant under the group of transformations $A_y: z_\nu\to y^{p_\nu}z_\nu$, $|y|=1$. A Cartan manifold is a complex manifold whose local coordinates map (suitable neighborhoods) topologically onto $(p_1, \cdots, p_n)$-domains such that these mappings are compatible with the action of the group of transformations A_y. Consequently, a Cartan manifold admits a 1-dimensional compact Lie group of complex automorphisms. The author proves that each complex manifold which admits an (infinite) compact Lie group of complex automorphisms is in fact a Cartan manifold.

H. Roehrl (Minneapolis, Minn.)

4080:

Arai, Hiraku. Remarks on holomorphic automorphisms of a simply-connected normal domain in several complex variables. Kōdai Math. Sem. Rep. **11** (1959), 88–94.

In der Arbeit werden Automorphismen einfach zusammenhängender Normalgebiete im n-dimensionalen komplexen Zahlenraum gewisse Automorphismen eines Produktraumes Riemannscher Flächen in eineindeutiger Weise zugeordnet. Dadurch lassen sich dann notwendige und hinreichende Bedingungen für die Starrheit solcher Gebiete angeben. {Es ist zu bemerken, daß Lemma 3 der Arbeit nicht in voller Allgemeinheit sondern nur für eine bestimmte Klasse von Normalgebieten richtig ist und folglich alle Sätze nur mit dieser Einschränkung gelten.}

H. Holmann (Münster)

4081:

Rothstein, Wolfgang. Bemerkungen zur Theorie komplexer Räume. Math. Ann. **137** (1959), 304–315.

Results of the theory of analytic surfaces are applied to the "continuation" of complex spaces. Let F be an analytic surface of complex dimension n "over" the complex projective space P^k $(k>n\geqq 2)$, let Z be a cycle of topological dimension $2n-1$ in F that separates F into two connected parts F_1 and F_2. Let Z be such that there exists an r-dimensional algebraic set $A\subset P^k$ such that no point of Z lies over A and such that $r+n\geqq k+1$ (trivial if Z is bounded). In earlier papers the author [Math. Ann. **122** (1951), 424–434; MR **12**, 818] and H. Sperling [Thesis, Marburg, 1957] have shown the following. (I) There exists an analytic surface F^* such that $F^*\supset F_1$ or $F^*\supset F_2$ and $\partial F^*=Z$, $F^*\cup Z$ is compact, and F^* has no points over A. (This process is called "inserting" an analytic surface into Z.) (II) Every function holomorphic (meromorphic) in a neighborhood $U(Z)$ of Z can be continued into F. (III) If Z is bounded, then every holomorphic mapping of a neighborhood $U(Z)$ into a complex number space that is nowhere degenerate in U can be continued to a holomorphic mapping of F^*, nowhere degenerate in F^*. Similar results hold for "half-cycles". (A half-cycle is for example that part of a cycle Z that is located over $|z_1|<1$, where $z_1, \cdots, z_k$ are the coordinates of P^k.) It is shown that the process of "inserting" analytic surfaces into cycles and half-cycles can be applied to analytically complex spaces. In particular the following results are obtained. For every holomorphically separable space K^n, $n\geqq 2$, there exists a complex space $'K^n\supseteq K^n$ such that all cycles that separate K^n into two connected subspaces are bounding. All functions holomorphic (meromorphic) continue to holomorphic (meromorphic) func-

tions in $'K^*$. If K^* is holomorphically convex, then $'K^* = K^*$. Similar results hold for holomorphically separable q-convex complex spaces (q-convex in the sense of the author). *H. J. Bremermann* (Berkeley, Calif.)

4082:

Rodrigues, A. A. Martins. Characteristic classes of complex homogeneous spaces. Bol. Soc. Mat. São Paulo **10** (1955), 67–86 (1958).

The author studies mainly the Chern classes of simply connected compact complex homogeneous spaces [called C-spaces in H.-C. Wang, Amer. J. Math. **76** (1954), 1–32; MR **16**, 518] along the following lines.

(1) The Chern classes of a compact complex manifold W can be obtained as the cohomology classes of certain differential forms which are defined by using an arbitrary connection and its curvature form in the principal bundle of complex frames of W with the complex general linear group as structure group. (2) In the case of a complex homogeneous space $W = K/H$, it is shown that there is a one-one correspondence between the set of invariant linear connections of type (1, 0) on K/H and the set of all complex bilinear functions α on $\bar{V} \times \bar{V}$ into $\bar{V}$ which are invariant by $\mathrm{ad}(H)$, where $\bar{V}$ is a subspace (provided with a natural complex structure) of the Lie algebra $L(K)$ of K such that $L(K) = \bar{V} + L(H)$ and $\mathrm{ad}(H)\bar{V} = \bar{V}$. The curvature form of any such connection can be expressed in terms of the corresponding function α. In particular, the connection corresponding to $\alpha = 0$ has curvature expressed by $\Theta(X, Y) = -\mathrm{ad}([X, Y]_H)$ acting on $\bar{V}$, where $X, Y \in \bar{V}$ and $[X, Y]_H$ denotes the $L(H)$-component. This part is an analogue of the results in the real homogeneous space [K. Nomizu, ibid. **76** (1954), 33–65; MR **15**, 468]. (3) Using the particular connection mentioned above, the expressions of differential forms representing the Chern classes of K/H are obtained in terms of $L(K) = \bar{V} + L(H)$ from the characteristic polynomial of $\Theta(X, Y)$. (4) Using the results of Wang on C-spaces, the endomorphism $\Theta(X, Y)$ is determined with respect to a suitable basis of $\bar{V}$ obtained from the root system.

From among the results obtained, we mention the following. If $W = K/X$ is a C-space (K a compact semisimple Lie group, X a C-subgroup of K), the Chern classes of W are zero in the dimensions greater than $\dim(K/X) - (\mathrm{rank}\, K - \mathrm{rank}\, X)$. A detailed study of characteristic classes of homogeneous spaces has appeared after the writing of the paper under review [see Borel and Hirzebruch, ibid. **80** (1958), 458–538; **81** (1959), 315–382; MR **21** #1586; **22** #988]. *K. Nomizu* (Providence, R.I.)

4083:

Lichnerowicz, André. Transformations analytiques d'une variété kählerienne compacte. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **2 (50)** (1958), 165–174.

Let V_{2n} be a compact Kähler manifold of real dimension $2n$; then it is also, of course, a Riemannian manifold. The author gives characterizations of infinitesimal conformal and isometric transformations of the Riemannian-Kähler structure and infinitesimal analytic transformations of the complex analytic structure—mostly in terms of the 1-form ξ which corresponds, relative to the metric, to the vector field X of the infinitesimal transformation—then using these he proves the theorem: If $n > 1$ [resp. $n = 1$] the largest connected group of conformal transformations of V_{2n} coincides with the largest connected group of analytic isometries [resp. analytic transformations] of the Kähler structure.

In addition the Lie algebra L_a of all analytic transformations is studied and special attention is given to the case of constant scalar curvature. Some of the results of this paper, including the theorem above, were announced earlier [C. R. Acad. Sci. Paris **244** (1957), 3011–3013; MR **20** #996]. *W. M. Boothby* (St. Louis, Mo.)

4084:

Goldberg, S. I. Groups of automorphisms of almost Kaehler manifolds. Bull. Amer. Math. Soc. **66** (1960), 180–183.

In this note the author announces results similar to those of A. Lichnerowicz [see preceding review], but for the more general case of almost Kähler manifolds; moreover, some results are obtained for Kähler and almost Kähler manifolds which do not require compactness. In particular, theorems obtained by the author are: (1) the theorem cited above of A. Lichnerowicz with Kähler replaced by almost Kähler; and (2) if the 1-form ξ corresponding to an infinitesimal conformal transformation X is closed and V_{2n} is a complete Kähler manifold ($n > 1$), then it is an infinitesimal automorphism of the Kähler structure—except that if the manifold is locally flat, then it is further required that the length of X be bounded.

W. M. Boothby (St. Louis, Mo.)

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